[FER]RELL'S ADVANCED ARITHMETIC

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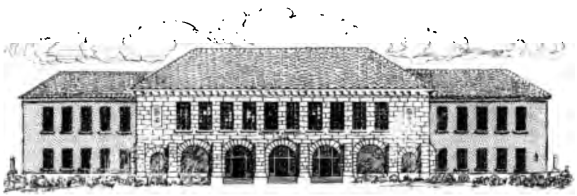

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FERRELL'S ARITHMETICS.

Ferrell's Teachers' and Students' Manual of Arithmetic.

165 Pages.

Ferrell's Elementary Arithmetic.

217 + xiii Pages.

Ferrell's Advanced Arithmetic.

408 + xvi Pages.

CRANE & COMPANY, PUBLISHERS, TOPEKA, KAN.

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PART I.

arithmetic. It includes the fundamental and derived processes of numbers expressed by figures and letters — integral and fractional, positive and negative. It gives all the operations of simple equations, and a limited discussion of quadratics. It classifies and discusses the nature of all kinds of arithmetical problems, and illustrates and explains the various methods of solution.

II. IT IS SCIENTIFIC.—If any one thought more than another has been an inspiration in the preparation of this book, it has been this: *There should be* SYSTEM *in studying and teaching arithmetic.*

Part I gives the pupil (1) a thorough and practical discussion of the fundamental and derived processes (except involution and evolution) of numbers expressed by figures, and (2) **one** *complete method* by equations and **one** by proportion for solving all the simpler arithmetical problems. There is not a new plan of reasoning and a new method of solution for each particular problem. Solutions are given complete. *This part contains the philosophical study of problems.*

Part II (1) gives the pupil such a knowledge of expressing numbers by letters as will enable him to understand and interpret a formula; (2) it teaches him how to classify problems into general types; and (3) how to develop and use the formulas for these types. *This part applies the science of arithmetic to the problems of practical business life.*

Part III gives the pupil (1) a knowledge of positive and negative numbers, (2) the method of solving more difficult problems — problems of one, two, three or more unknown numbers, and (3) a limited knowledge of quadratic equations and pro-

gressions. *This part prepares the pupil for the study of higher mathematics.*

III. It is Teachable.—The teacher and the pupil are not left to guess at the author's methods of presenting the various subjects. Every part is explained and illustrated, and numerous examples are given. The pupil always has before him a model to study and follow, and there is no excuse for his efforts being haphazard and aimless. The plan of the book is adapted to the developing mind both in *arrangement* and *explanations*.

For suggestions on teaching this book, see Introduction.

J. A. F.

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INTRODUCTION.

To the Teacher:

I. Plan.—Our best geometries first solve and give full explanations of all the fundamental problems and propositions of a book (or part) of geometry for the pupil's study; and then give at appropriate places or at the close of the part a list of problems and propositions for the pupil to solve. The logic of this plan appears in this: That the pupil is taught the nature of problems and correct methods of solving them before he is required to solve them for himself.

This is in general the author's plan of teaching not only geometry, but all the branches of mathematics, *especially all of the lower branches.*

Note.—It is bad pedagogy, contrary to business judgment, and a great waste of time, to put a pupil at work trying to solve problems before he has a knowledge of the nature of such problems as he is expected to solve, or a knowledge of the plan by which such problems are to be solved.

Throughout this book the matter is arranged in two divisions or parts: *explanatory part*, and *exercises*. The *explanatory part* consists of definitions, principles, rules, explanations, and examples of processes and solutions. This part is to be *learned* by the pupil. The *exercises* are to be *worked out* by the pupil.

II. How to Use the Explanatory Part.—*Assign explanatory matter for study and recitation.* (1) At the time of assigning a lesson, go over it with the pupil, and explain such (and only such) things as, in your judgment, your pupil will be unable to understand for himself. (2) Require pupils to commit definitions and principles to memory.

Note.—Do not tolerate "parrot" memorizing. Before the pupil attempts to commit a definition, he should find out what it means. Half

of the work of memorizing is done when the pupil understands the meaning of the language used.

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NOTE.—It is the least of the author's intentions to permit the pupil to pass over *examples* as something to which he *may* refer when in trouble—something to be studied or neglected at his pleasure. A thorough understanding of the example comes first in time and importance in studying and teaching this book.

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IV. ADDITIONAL EXERCISES.—Exercise LVI has only 26 problems. Nearly all the problems in Exercises CIX to CXVI are percentage problems, and the pupil should be able to solve any of them by the method given in Article 80. It is recommended that the pupil be given a thorough knowledge of the solution of percentage problems by the *equation method*.

V. ANSWERS.—Answers are not put in this book. The author believes that the presence of *answers*, either with the problems or at the close of the book, fosters dependency and lack of self-confidence in the pupil. For teachers, who may need answers to save time and labor in correcting and grading school work, the *answers* are printed in a pamphlet, and may be obtained from the publishers.

ADVANCED ARITHMETIC.

PART I.

I. STUDY OF NUMBERS.

A. INTRODUCTION.

1. Definitions.—Arithmetic is the introductory branch of the science of numbers and problems.

Science is classified knowledge.

A **Number** is one or more units, considered as forming one quantity, or amount. An *expression* of one or more units is also called a number. The word *number* is a term signifying one or more units, or an expression of one or more units.

NOTE.—A distinction should be made between "*number*" as used in the sentence, *A number of apples were eaten,* where the word *number* means quantity or amount; and "*number*" as used in the sentence, *The divisor is the number on the left,* where the word *number* means the expression on paper.

A **Unit** is a single thing, or one. Units are classed as *concrete* or *abstract*, *integral* or *fractional.*

A **Concrete Unit** is one *thing*. As,

1 book, 1 man, 1 box.

An **Abstract Unit** is simply *one*, a ratio, and not a thing. As,

A unit, not considered as a part of any other unit, is an **Integral Unit.** As,

1, 1 book, 1 foot.

A unit, considered as a part of some other unit, is a **Fractional Unit.** As,

$\frac{1}{5}$ foot, $\frac{1}{4}$ dollar, $\frac{1}{8}$ pint.

NOTE.—The $\frac{1}{5}$ is 1 fifth of 1 foot. Then, the unit, $\frac{1}{5}$, is a part of the unit, 1 foot; so, the $\frac{1}{4}$ is a part of $1 and the $\frac{1}{8}$ is a part of 1 pint. The unit of which the fractional unit is a part is called the *Unit of the Fraction.*

A number composed of concrete units is a **Concrete Number.** As,

6 men, 15 dollars, 20 miles.

A number composed of abstract units is an **Abstract Number.** As,

6, 15, 20.

When a concrete number and an abstract number have each the same number of units, they are called *Corresponding* numbers. *Every concrete number has its corresponding abstract number.*

NOTE.—This fact will be important, when you come to solve problems by proportion and by formulas.

A **Simple Number** is a number whose integral units are of the same *kind* and *size.* As,

250 bushels, 75 men.

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5 yards 2 feet; 4 bushels 2 pecks 5 quarts.

A number composed of integral units is an **Integral Number,** or an **Integer.** As,

12 men, 25.

NOTE.—An integer may be either concrete or abstract.

A number composed of fractional units is a **Fractional Number,** or a **Fraction.** As,

$\frac{3}{5}$ dollar, $\frac{5}{13}$.

NOTE.—A fractional number may be either concrete or abstract.

A number composed partly of integral units and partly of fractional units is a **Mixed Number.** As,

$5\frac{1}{2}$ dollars, $6\frac{2}{3}$.

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Find the cost of 10 books, at 30¢ each.

But such problems have all the essentials of questions, and may be easily so stated. Thus,

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1. What is *arithmetic?*
2. Are there other branches of the science of number?
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4. What is *science?*
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7. What is *quantity?*
8. What is a *unit?*

9. In the sentence, The number on the right is the quotient, does the word number signify one or more units or does it signify the *expression* that is written?

10. In the sentence, I do not know the number of marbles that were lost, is the word *number* and the real number of marbles the same thing? What is the difference, if any?

11. In the sentence, There are 7 books on the table, is the expression, "7 books," and the real number of books the same thing? What is the difference, if any?

12. Number is sometimes defined as a collection of units. Do the units have to be collected, or may they be considered together as forming one quantity without being in reality collected? Illustrate your answer with an example.

13. Give the classification of units.

14. What is a *concrete unit?* Give an example.

15. What is an *abstract unit?* Give an example.

16. What is an *integral unit?* Give an example.

17. What is a *fractional unit?* Give an example.

18. What is a *concrete number?* Give an example.

19. What is an *abstract number?* Give an example.

20. What are *corresponding numbers?* Give an example.

21. What is an *integer?* Give an example.

22. What is a *fraction?* Give an example.

23. What is the *unit of a fraction?* Give an example.

24. What is a *mixed number?* Give an example.

25. What is a *problem?*

26. Is a problem always in the form of a question? Illustrate your answer by examples.

27. Are all questions problems? Illustrate your answer by examples.

B. NOTATION AND NUMERATION OF INTEGERS.

2. Definitions.—Notation is the process of expressing numbers by means of words, letters, or figures.

Numeration is the process of reading numbers, expressed in words, letters, or figures.

NOTE.—A pupil that can correctly write numbers can read them, for the same knowledge of the method, of use of terms and characters, is required in both. Hence, both processes are treated together in this book.

3. Methods.—Five methods of expressing numbers are in common use:

(1) By means of words that express the number of units considered. As,

three, nine, twenty-five.

(2) By means of letters that express the number of units considered. As,

III, IX, XXV.

NOTE.—Neither of the above methods is used in arithmetical computations. Their study belongs to the reader rather than to the arithmetic.

(3) By means of words that do not express the number of units considered. As,

cost price, John's age, B's money.

NOTE.—This method of expressing number is used in the solution of arithmetical problems. It is self-explanatory, and does not require special study at this time.

(4) By means of letters that do not express the number of units considered. As,

x, $2\,ay$, m^2.

NOTE.—This method is usually called *algebraic notation*. It is studied and used in Parts II and III.

(5) By means of figures that express the number of units considered. As,

3, 9, 25.

NOTE.—This method as herein used is called the *Arabic method*. It is the common method of expressing known numbers in arithmetical computations.

4. Figures.—The Arabic method of notation employs ten characters, called **Figures** or **Digits.** Their forms and names are as follows:

0,	1,	2,	3,	4,	5,	6,	7,	8,	9.
zero,	one,	two,	three,	four,	five,	six,	seven,	eight,	nine.

NOTE.—The 0 is also called *naught.* Some authors claim that there are only nine *digits*, and that 0, representing no value, is not a *digit.*

5. Reading and Writing Integers to One Thousand.—In reading and writing numbers by the Arabic method—

(1) The integers up to ten are named and written with figures as follows:

one,	1	five,	5
two,	2	six,	6
three,	3	seven,	7
four,	4	eight,	8

nine, 9

(2) In integers of more than nine units, the units are grouped as far as possible into groups of ten each. Groups of tens up to ten tens are named and written with figures as follows:

1 ten,	ten,	10	5 tens,	fifty,	50
2 tens,	twenty,	20	6 tens,	sixty,	60
3 tens,	thirty,	30	7 tens,	seventy,	70
4 tens,	forty,	40	8 tens,	eighty,	80

9 tens, ninety, 90

(3) In integers of 10 tens or more, the groups of ten each are grouped as far as possible into larger groups of 10 tens each. One of these larger groups is called a *hundred.* Hun-

dreds up to 10 hundreds are named and written with figures as follows:

one hundred,	100	five hundred,	500
two hundred,	200	six hundred,	600
three hundred,	300	seven hundred,	700
four hundred,	400	eight hundred,	800

nine hundred, 900

(4) Integers between 10 and 20 are named and written with figures as follows:

1 ten 1 unit;	eleven;	11
1 ten 2 units;	twelve;	12
1 ten 3 units;	thirteen;	13
1 ten 4 units;	fourteen;	14
1 ten 5 units;	fifteen;	15
1 ten 6 units;	sixteen;	16
1 ten 7 units;	seventeen;	17
1 ten 8 units;	eighteen;	18
1 ten 9 units;	nineteen;	19

(5) Any integer between 20 and 100 is written with figures by placing the number of tens in the second place from the right or *tens' place* and the number of units in the right-hand place or *units' place*. If there are no units, place a 0 in units' place. These numbers are read by naming the tens and units in succession. Thus:

21,	twenty-one	47,	forty-seven
78,	seventy-eight	86,	eighty-six
55,	fifty-five	34,	thirty-four
92,	ninety-two	63,	sixty-three

99, ninety-nine

(6) Any integer between 1 hundred and 10 hundred is written by placing the number of hundreds in the third place from

the right, or *hundreds' place;* the number of tens in tens' place; and the number of units in units' place. Fill vacant places with 0's. These numbers are read by naming the hundreds, tens, and units in succession. Thus:

125, one hundred twenty-five
790, seven hundred ninety
467, four hundred sixty-seven
806, eight hundred six
512, five hundred twelve
272, two hundred seventy-two
638, six hundred thirty-eight
301, three hundred one
999, nine hundred ninety-nine

NOTE.—Do not use "*and*" in reading these numbers.

Since, **1 hundred = 10 tens,** and
1 ten = 10 units,

a figure in hundreds' place represents *10 times* as many as the same figure in tens' place; a figure in tens' place represents *10 times* as many as the same figure in units' place.

EXERCISE II.

Read rapidly:

1.	*2.*	*3.*	*4.*	*5.*
78	34	800	065*	259
19	60	139	315	734
25	71	240	725	813
82	88	327	899	004*
96	07*	444	863	117
57	56	516	992	629
43	99	607	403	999

* In integers, a 0 standing on the left of other figures has no effect upon the number. 07 is 7. 065 is 65. 004 is 4.

Write:

6.	*7.*	*8.*	*9.*
twenty-one	fifty-four	thirty-nine	sixty-nine
seventy	ninety	eighty-one	thirty-seven
eighty-nine	seventy-two	thirteen	ninety-three
thirty-eight	nineteen	seventy-three	forty-five
forty-six	sixty-six	ninety-eight	eighty-four

10.	*11.*
Four hundred ninety-four	Eight hundred seventy-two
Two hundred nineteen	Three hundred thirty-three
Five hundred fifty-five	One hundred fourteen
Seven hundred seventy-nine	Nine hundred three
Eight hundred six	Eight hundred seventy-six
Nine hundred sixty-nine	Six hundred twenty-four
Eight hundred sixty	Five hundred nineteen
Seven hundred ten	Seven hundred ninety-four
Six hundred sixty-six	Two hundred forty-eight
Four hundred	Nine hundred ninety-nine

6. Notation and Numeration of Integers in General.—By the Arabic method, integers larger than nine are represented by figures placed side by side. Thus:

274139568

The *8* represents 1's, or units of the **first order**;
The *6* represents 10's, or units of the **second order**;
The *5*, units of the **third order**;
The *9*, units of the **fourth order**;
and so on.

The *order* of any figure or unit is the number of its place, counting from the right.

This method of representing integers is based upon the following law:

LAW.—*Ten units of any one order make one unit of the next higher order.*

In reading and writing integers, the figures are considered, as far as possible, in groups of three figures each, commencing at the right hand. These groups are called **Periods.** The first twelve periods, counting from the right, are named as follows:

units	quintillions
thousands	sextillions
millions	septillions
billions	octillions
trillions	nonillions
quadrillions	decillions

NOTE.—If the word, *period,* follows the name, the possessive sign should be used. Either *period of thousands* or *thousands' period* is correct.

Let it be required to read the number,

346524207985325.

Separating it into periods, and naming the periods, we have—

trillions,	*billions,*	*millions,*	*thousands,*	*units.*
346,	524,	207,	985,	325.

How many units? *Ans.*, 325.

How many thousand? *Ans.*, 985.

How many million? *Ans.*, 207.

How many billion? *Ans.*, 524.

How many trillion? *Ans.*, 346.

Then read the number, beginning with the left-hand period. Thus, 346 trillion, 524 billion, 207 million, 985 thousand, 325 (units).

NOTE.—In reading numbers, drop the word *units*, and say *hundred, thousand, million,* &c., not *hundreds, thousands, millions,* &c. Never use "*and*" in reading integers.

The figures in each period are read by methods explained in Article 5; then the name of the period is added.

Let it be required to write *seventy-five billion, nine hundred forty-eight million, five hundred one thousand, three hundred sixty-two.*

How many billion ? *Ans.*, 75.
How many million ? *Ans.*, 948.
How many thousand ? *Ans.*, 501.
How many units ? *Ans.*, 362.
Then write the number. Thus, 75,948,501,362.

NOTE.—When it is desired to separate the periods of a number, *commas* are used; as in the last number above.

Since a period has three places, the place on the right hand is *units* of the period, the second place is *tens* and the third place is *hundreds*. To illustrate, the first place in thousands is units of thousands, or simply *thousands;* the second place, *ten-thousands;* the third place, *hundred-thousands*. The first place in millions is *millions;* the second, *ten-millions;* the third, *hundred-millions;* and so on.

The method of grouping three figures into a period, as explained above, is the method used by the people of the United States and France, and is called the **French Method.** The English people and people of many other European nations consider a period as made up of six figures. This method of grouping the figures is called the **English Method.** The first six periods according to this method are named as follows:

units	trillions
millions	quadrillions
billions	quintillions

Let it be required to read 346524207985325 by the English method.

Separating the number into periods, we have —

billions,	*millions,*	*units.*
346,	524207,	985325

How many units? *Ans.*, 985325.
How many million? *Ans.*, 524207.
How many billion? *Ans.*, 346.

Then read the number, beginning with the left-hand period. Thus, 346 billion, 524207 million, 985325 (units).

NOTE.—The figures in each period are read by the methods already explained, then the name of the period is added.

EXAMPLES.

1. Read 3462465075.

Process : 3,462,465,075; 3 billion, 462 million, 465 thousand, 75.

NOTE.— Use the French method unless the English method is called for or indicated. Commas are not commonly used to separate the periods, unless there are three or more periods. In smaller numbers, the periods are located mentally.

2. Read 672000024941.

Process : 672,000,024,941; 672 billion, 24 thousand, 941.

3. Read 873002403400540 (English method).

Process : 873,002403,400540; 873 billion, 2403 million, 400540.

4. Write 894 billion, 1 thousand, 127.

Written : 894,000,001,127.

NOTE.—There are no millions; fill the orders with 0's. There are no hundreds of thousands or tens of thousands; fill these orders with 0's.

5. Write 1753 billion, 201400 million, 843004.

Written : 1753,201400,843004.

6. Write 5 units of the 6th order, 7 units of the 3d order, 1 unit of the 2d order, 8 units of the 1st order.

Written: 500718.

NOTE.—Fill the 5th and 4th orders with 0's.

EXERCISE III.

1. What is the law upon which the Arabic method of reading and writing numbers is based?

2. What is the order of a figure? How are orders numbered?

3. How many figures in a period according to the French method? Where do you commence to point off a number into periods?

4. How many figures in a period according to the English method?

5. Name in order twelve periods of the French method.

6. Name in order six periods of the English method.

7. Do the French and English methods both conform to the law of the Arabic method?

Read by the French method:

8. 43250
9. 678426
10. 9407081
11. 5675694073
12. 889970708432
13. 78604501257943
14. 10030040250641
15. 83000034562135
16. 786429536742
17. 132103243576842

Read by the English method:

18. 765302456901327.
19. 1234567878785321.
20. 986435200340561256.

Write:

21. 55 thousand, 421.
22. 704 million, 73 thousand, 25.

23. 846 billion, 721 million, 1 thousand.

24. Twenty-five thousand seven hundred one.

25. Seventy-six billion, eight hundred one million, sixteen thousand three hundred seven.

26. Seven units of the 5th order, nine units of the 4th order, six units of the 3d order, one unit of the 2d order, 8 units of the 1st order.

27. 8 units of the 7th order, 2 units of the 5th order, 5 units of the 4th order, 9 units of the 1st order.

28. 125 quadrillion, 125 trillion, 125 million, 125.

29. Five hundred sixty-six thousand two hundred eighty-one million, eighty-nine thousand twenty-one.

30. 675421 billion, 604001 million, 345762.

7. Reading and Writing Dollars and Cents.— The *dollar-mark*, $, is always placed before the number with which it is used. Thus,

$5, five dollars.
$908, nine hundred eight dollars.

Cents may be expressed with dollars or with a dollar-mark. When so expressed, *two places* are always used for cents.

100 cents=1 dollar; then,

any number of cents less than a dollar may be written with two figures, of which the right-hand figure is for the units of cents and the left-hand figure is for the tens of cents (dimes). A period (.) is always placed between dollars and cents and the word *and* is used there in reading. Thus,

$5.10, 5 dollars and 10 cents;
$12.06, 12 dollars and 6 cents;
$.05, 5 cents;
$807.50, 807 dollars and fifty cents.

Cents may be written by using the *cent-mark*, ¢. Thus,

3¢, 3 cents; 20¢, 20 cents; 85¢, 85 cents.

EXERCISE IV.

Read:

1.	*2.*	*3.*	*4.*
$7	$10.10	$.40	25¢
$59	$72.55	$.75	5¢
$408	$246.05	$.03	70¢
$3906	$109.13	$.99	99¢

5. Write with $: Two dollars and forty cents; seventeen dollars and seventy-five cents; nine thousand five hundred sixty-three dollars and twenty-five cents; two cents; fifty cents.

6. Write with ¢: Twenty-five cents; nine cents; eighteen cents; one hundred twenty-five cents; ninety-five cents.

C. ADDITION OF INTEGERS.

8. Definitions, Signs, and Principles.—Addition is the process of uniting two or more numbers into one. The numbers to be added are called **Addends.** The result of addition is called the **Sum** or **Amount.**

The **Sign of Addition,** +, is read *plus.*

The **Sign of Equality,** =, is read *equals.*

$$17+9=26$$

is read *17 plus 9 equals 26,* and means that 9 added to 17 gives a result of 26. *17* and *9* are *addends,* and *26* is the *sum.*

In the fundamental processes of mathematics, certain self-evident *principles* must be observed. In addition they are as follows:

PRINCIPLES: *1. Only similar numbers can be added.*

2. The sum contains all the units of all addends.

3. The sum and the addends must be similar numbers.

4. The sum is the same, regardless of the order in which the numbers are added.

NOTE.—*Similar numbers*, as here used, are those whose integral units are of the same kind and size.

9. Process of Addition.—Write the addends so that figures of the same order will stand in a column. Always begin with units to add.

Ten units of one order make one of the next higher order. Then, in adding numbers of more than one order, if the sum of all the units of any order is ten or more, the tens of that order are considered as units of the next higher order, and are so added.

EXAMPLE.

Add 576, 342, 75, 981, 203.

Process.

576
342
75
981
203
2177, result.

Explanation: (1) The sum of the units of the first column (the one on the right) is 17, or 1 unit of the 2d order and 7 units of the first order. Write the *7* below, and carry the *1* to the next column.

(2) The sum of the units of the 2d column, including the *1* carried, is 27, or 2 units of the 3d order and 7 units of the 2d order. Write the *7* below and carry the *2* to the next column.

(3) The sum of the units of the 3d column, including the *2* carried, is 21, or 2 units of the 4th order and 1 unit of the 3d order. Write the *1* and the *2* below in their proper orders.

TESTS OF ACCURACY: (1) *Review the work, or* (2) *add the columns in reverse order.*

NOTE.—Various other tests of accuracy in addition are suggested by different authors. Such as, casting out the 9's, or by separating the addends into two or more groups, adding each group, then adding their sums. Such methods are not resorted to once in a thousand times to test the accuracy of addition.

EXERCISE V.

Copy and find sums:

1.	*2.*	*3.*	*4.*	*5.*	*6.*
346	766	555	276	321	482
425	675	462	549	374	888
879	578	888	695	999	476
276	989	951	598	846	327
942	876	377	846	594	348
379	768	894	589	666	555
566	898	856	444	389	776

7.	*8.*	*9.*	*10.*	*11.*	*12.*
942	372	584	829	938	456
224	884	382	559	986	472
337	724	486	629	729	829
246	654	785	473	259	464
461	542	632	545	454	923
729	421	428	882	793	846
288	899	987	765	455	999

13.	*14.*	*15.*	*16.*	*17.*
12467	99999	95487	57684	83579
86492	86897	58987	66666	54865
94876	58987	89987	57868	47986
87594	68859	66666	47984	55555
59187	45958	97869	38946	84484
67895	73986	77777	98979	98897
78957	54852	12345	78787	48864
98798	77958	99999	57978	87758
78978	68879	98765	86868	69589
67594	59987	88888	69699	57868
98979	68827	48767	37379	48573
64888	77989	56789	87654	48452

18.	*19.*	*20.*
123456789	876543219	24678
547695437	34768492	579642
865796543	99999	8978687
759876987	7586493	59875879
666666666	59876538	4769876543
989898989	789437598	989898989
465798788	888888888	65943756
543543543	958764356	987689
999988888	579867958	8976576438
789789789	576576576	9578495789

D. SUBTRACTION OF INTEGERS.

10. Definitions, Sign, and Principles.—Subtraction is the process of taking one number from another. The number taken is called the **Subtrahend.** The number from which the subtrahend is taken is called the **Minuend.** The result of subtraction is called the **Remainder** or **Difference.**

The **Sign of Subtraction,** –, is read *minus.*

Thus, $17-9=8$

is read *17 minus 9 equals 8,* and means that 9 taken from 17 leaves 8. *17* is the *minuend; 9,* the *subtrahend;* and *8,* the *remainder.*

PRINCIPLES: *1. The minuend, subtrahend, and remainder must be similar numbers.*

2. The remainder equals the minuend minus the subtrahend.

3. The subtrahend equals the minuend minus the remainder.

4. The minuend equals the sum of the subtrahend and remainder.

11. Process of Subtraction.— Place the subtrahend under the minuend, units under units, tens under tens, etc. Always begin with units to subtract.

One of any order (except the first order) makes ten of the next

lower order. When a figure of the minuend represents fewer units than the corresponding figure of the subtrahend, one unit is taken ("borrowed") from the next higher order of the minuend and considered as ten of the lower order.

EXAMPLES.

1. From 43 take 26.

Process.

$$\begin{array}{r} 43 \\ \underline{26} \\ 17 \end{array}$$

Explanation complete: (1) We cannot take 6 units from 3 units; but we can take one ten of the 4 tens, and add it to the 3 units. This will make 13 units. Then, 13 units − 6 units = 7 units. Write the *7* below. (2) One ten has been taken from the 4 tens, which leaves but 3 tens in the minuend. 3 tens − 2 tens leaves 1 ten. Write the *1* below.

2. From 704 take 375.

Process.

$$\begin{array}{r} 704 \\ \underline{375} \\ 329 \end{array}$$

Explanation complete: (1) We cannot take 5 units from 4 units, and there are no tens in the minuend; then we must take 1 hundred from the 7 hundred: 1 hundred = 10 tens. Now, we can take 1 of the 10 tens and add it to the 4 units. 1 ten + 4 units = 14 units. 14 units − 5 units = 9 units. Write the *9* below.

(2) Of the 10 tens we took from hundreds' place, we have used 1 ten and have 9 tens left. 9 tens − 7 tens = 2 tens. Write the *2* below.

(3) We have used 1 hundred of the 7 hundred; then we have only 6 hundred left. 6 hundred − 3 hundred = 3 hundred. Write the *3* below.

3. From 8801 take 5674.

Process.

$$\begin{array}{r} 8801 \\ \underline{5674} \\ 3127 \end{array}$$

Explanation shortened: (1) Looking upon *1*, think 11 (why ?). 11 − 4 = 7. Write the *7* below.

(2) Looking upon the *0*, think 10, then 9 (why ?). 9 − 7 = 2. Write the *2* below.

(3) Looking upon the *8*, think 7 (why ?). 7 − 6 = 1. Write the *1* below.

(4) 8 − 5 = 3. Write the *3* below.

TESTS OF ACCURACY: (1) *Review the work;* or (2) *the minuend minus the remainder should equal the subtrahend;* or (3) *the sum of the subtrahend and remainder should equal the minuend.*

EXERCISE VI.

Copy and find remainders:

1.	*2.*	*3.*	*4.*	*5.*
764	591	932	846	821
178	486	763	487	295
6.	*7.*	*8.*	*9.*	*10.*
837	700	500	800	400
639	583	479	365	123
11.	*12.*	*13.*	*14.*	*15.*
8426	7363	8342	1264	1348
7544	5486	486	947	796
16.	*17.*	*18.*	*19.*	*20.*
1920	1930	2004	2245	6472
749	806	1759	1892	2803
21.	*22.*	*23.*	*24.*	*25.*
7038	8333	1675	2008	3004
5984	2345	597	1009	2106

Find remainders:

26. 94675 − 67897
27. 58406 − 57643
28. 70698 − 67899
29. 54863 − 43794
30. 60054 − 54376
31. 92345 − 56789
32. 80125 − 43056
33. 72010 − 35476
34. 82904 − 68926
35. 92500 − 47832
36. 76543 − 34567
37. 345678 − 95876
38. 193846 − 87909
39. 150000 − 74567
40. 183456 − 89999
41. 100000 − 65432
42. 240000 − 186426
43. 170984 − 145096
44. 333333 − 273747
45. 811221 − 456789
46. 830405 − 640506
47. 900002 − 712345

E. MULTIPLICATION OF INTEGERS.

12. Definition, Sign, and Principles.—Multiplication is a process, shorter than addition, for finding the sum when one number is to be used as an addend several times. The number to be used as the addend is called the **Multiplicand.** The number showing how many times the multiplicand is to be used is called the **Multiplier.** The result of multiplication is called the **Product.** The expression, "*multiplied by*," used to indicate multiplication, means that the number placed *before* it is to be used as the *multiplicand* and the number placed *after* it is to be used as the *multiplier.*

The expression, "*multiplied together*," is used to indicate the multiplication of one number by another without designating which number is to be used as the multiplier or multiplicand.

The **Sign of Multiplication,** ×, is read *times.** "*1*×" is read *one time*, or *once;* "*2*×," *two times*, or *twice.*

$$5 \times 7 = 35$$

is read *5 times 7 equals 35,* and means that the sum obtained by using 7 as an addend 5 times is 35. The *5* is the *multiplier;* the *7*, the *multiplicand;* and the *35*, the *product.*

PRINCIPLES: *1. The multiplicand may be either concrete or abstract.*

2. The multiplier must be abstract.

3. The product must be similar to the multiplicand.

4. When the two numbers to be multiplied together are both abstract, the product is the same, whichever number is taken as the multiplier.

NOTE.—Before proceeding with multiplication, the pupil should have a thorough knowledge of the *multiplication table.*

---

*Many authors also read the sign, ×, *multiplied by.* In that reading, the multiplicand must come before and the multiplier after the sign.

For reasons which it would not be profitable to discuss here, the author prefers to read the sign *times*, and adheres to that reading throughout this book.

MULTIPLICATION TABLE.

1 × 0 = 0	2 × 0 = 0	3 × 0 = 0	4 × 0 = 0
1 × 1 = 1	2 × 1 = 2	3 × 1 = 3	4 × 1 = 4
1 × 2 = 2	2 × 2 = 4	3 × 2 = 6	4 × 2 = 8
1 × 3 = 3	2 × 3 = 6	3 × 3 = 9	4 × 3 = 12
1 × 4 = 4	2 × 4 = 8	3 × 4 = 12	4 × 4 = 16
1 × 5 = 5	2 × 5 = 10	3 × 5 = 15	4 × 5 = 20
1 × 6 = 6	2 × 6 = 12	3 × 6 = 18	4 × 6 = 24
1 × 7 = 7	2 × 7 = 14	3 × 7 = 21	4 × 7 = 28
1 × 8 = 8	2 × 8 = 16	3 × 8 = 24	4 × 8 = 32
1 × 9 = 9	2 × 9 = 18	3 × 9 = 27	4 × 9 = 36
1 × 10 = 10	2 × 10 = 20	3 × 10 = 30	4 × 10 = 40
1 × 11 = 11	2 × 11 = 22	3 × 11 = 33	4 × 11 = 44
1 × 12 = 12	2 × 12 = 24	3 × 12 = 36	4 × 12 = 48
5 × 0 = 0	6 × 0 = 0	7 × 0 = 0	8 × 0 = 0
5 × 1 = 5	6 × 1 = 6	7 × 1 = 7	8 × 1 = 8
5 × 2 = 10	6 × 2 = 12	7 × 2 = 14	8 × 2 = 16
5 × 3 = 15	6 × 3 = 18	7 × 3 = 21	8 × 3 = 24
5 × 4 = 20	6 × 4 = 24	7 × 4 = 28	8 × 4 = 32
5 × 5 = 25	6 × 5 = 30	7 × 5 = 35	8 × 5 = 40
5 × 6 = 30	6 × 6 = 36	7 × 6 = 42	8 × 6 = 48
5 × 7 = 35	6 × 7 = 42	7 × 7 = 49	8 × 7 = 56
5 × 8 = 40	6 × 8 = 48	7 × 8 = 56	8 × 8 = 64
5 × 9 = 45	6 × 9 = 54	7 × 9 = 63	8 × 9 = 72
5 × 10 = 50	6 × 10 = 60	7 × 10 = 70	8 × 10 = 80
5 × 11 = 55	6 × 11 = 66	7 × 11 = 77	8 × 11 = 88
5 × 12 = 60	6 × 12 = 72	7 × 12 = 84	8 × 12 = 96
9 × 0 = 0	10 × 0 = 0	11 × 0 = 0	12 × 0 = 0
9 × 1 = 9	10 × 1 = 10	11 × 1 = 11	12 × 1 = 12
9 × 2 = 18	10 × 2 = 20	11 × 2 = 22	12 × 2 = 24
9 × 3 = 27	10 × 3 = 30	11 × 3 = 33	12 × 3 = 36
9 × 4 = 36	10 × 4 = 40	11 × 4 = 44	12 × 4 = 48
9 × 5 = 45	10 × 5 = 50	11 × 5 = 55	12 × 5 = 60
9 × 6 = 54	10 × 6 = 60	11 × 6 = 66	12 × 6 = 72
9 × 7 = 63	10 × 7 = 70	11 × 7 = 77	12 × 7 = 84
9 × 8 = 72	10 × 8 = 80	11 × 8 = 88	12 × 8 = 96
9 × 9 = 81	10 × 9 = 90	11 × 9 = 99	12 × 9 = 108
9 × 10 = 90	10 × 10 = 100	11 × 10 = 110	12 × 10 = 120
9 × 11 = 99	10 × 11 = 110	11 × 11 = 121	12 × 11 = 132
9 × 12 = 108	10 × 12 = 120	11 × 12 = 132	12 × 12 = 144

13. Process of Multiplication.—Place the multiplier under the multiplicand; multiply by each figure of the multiplier, beginning with units. Always place the right-hand figure of each product under that figure of the multiplier used in obtaining it. Add these products for the *complete product.*

EXAMPLES.

1. Multiply 379 by 67.

Process.

$$\begin{array}{r} 379 \\ \underline{67} \\ 2653 \\ \underline{2274} \\ 25393 \end{array}$$, result.

Explanation: (1) $7\times9=63$. Write the *3* below and carry the *6*. $7\times7=49$. $49+6$ (carried) $=55$. Write the *5* (units) below, and carry the *5* (tens). $7\times3=21$. $21+5$ (carried) $=26$. Write the *26* below.

(2) Multiply by the *6* in the same way.

(3) Add these products. The complete product is 25393.

NOTE.—The *6* of the multiplier is 6 tens, and the first figure of its product, which is 4, is also 4 tens. Therefore the *4* is placed under the *6*, or in the second order.

NOTE.—The "Long Way" given in the following examples is for explanation, and not to be practiced by the pupil.

2. Multiply 57 by 40.

Long Way.

$$\begin{array}{r} 57 \\ \underline{40} \\ 00 \\ \underline{228} \\ 2280 \end{array}$$

NOTE.—In the *Short Way*, we place the *4* under the 7, and the *0* to the right of the 4. Bring down the *0* and multiply by *4* only.

Short Way.

$$\begin{array}{r} 57 \\ \underline{40} \\ 2280 \end{array}$$

3. Multiply 370 by 500.

Long Way.

$$\begin{array}{r} 370 \\ \underline{500} \\ 000 \\ 000 \\ \underline{1850} \\ 185000 \end{array}$$

NOTE.—In the *Short Way*, we place the *5* under the 7, letting the 0's fall to the right. Multiply by the *5*, and bring down three 0's —one for the 0 in the multiplicand, and two for the 0's in the multiplier.

Short Way.

$$\begin{array}{r} 370 \\ \underline{500} \\ 185000 \end{array}$$

4. Multiply 246 by 502.

Long Way.

```
   246
   502
  ----
   492
  000
 1230
------
123492
```

NOTE.—We need not multiply by the 0 in the multiplier. But we must be careful to put the *0* of the partial product obtained by multiplying by the *5* under the *5*.

Short Way.

```
   246
   502
  ----
   492
 1230
------
123492
```

5. Multiply 678 by 1200.

```
   678
  1200
------
813600
```

NOTE.—Multiply by 12, not by 2 and 1.

TESTS OF ACCURACY: (1) *Review the work*, or (2) *multiply the multiplier by the multiplicand, considering the multiplicand abstract.*

EXERCISE VII.

Find products:

1. 400×583*	*16.* 724×6743
2. 530×475	*17.* 594×3948
3. 640×873	*18.* 475×9873
4. 370×582	*19.* 649×8479
5. 420×897	*20.* 709×4377
6. 730×948	*21.* 806×9835
7. 890×345	*22.* 639×3829
8. 900×760	*23.* 738×5893
9. 730×800	*24.* 1263×85643
10. 1200×548	*25.* 2095×56789
11. 1100×675	*26.* 4075×83064
12. 246×375	*27.* 8764×93725
13. 856×948	*28.* 4609×85274
14. 987×987	*29.* 5007×95683
15. 876×955	*30.* 4199×87878

* The multiplier is placed before and the multiplicand after the sign "×."

31. Multiply 504763 by 943000
32. Multiply 724256 by 23462
33. Multiply 958740 by 37542
34. Multiply 818475 by 30724
35. Multiply 798643 by 48056
36. Multiply 829374 by 58376
37. Multiply 941072 by 86754
38. Multiply 430762 by 10000
39. Multiply 708340 by 25000
40. Multiply 984632 by 92345

41. $75 \times 94 \times 237$
42. $184 \times 276 \times 329$
43. $101 \times 279 \times 834$
44. $642 \times 200 \times 730$
45. $820 \times 540 \times 422$
46. $25 \times 34 \times 63 \times 75$
47. $38 \times 240 \times 320 \times 97$
48. $39 \times 800 \times 600 \times 240$
49. $42 \times 560 \times 720 \times 980$
50. $125 \times 336 \times 457 \times 789$

F. DIVISION OF INTEGERS.

14. Definitions, Signs, and Principles.—Division is the process of finding how many times one number contains another. The number contained is called the **Divisor.** The number containing the divisor is called the **Dividend.** The result of division is called the **Quotient.**

The **Sign of Division,** ÷, is read *divided by.* The **Sign of Ratio,** :, which is also a sign of division, is read *the ratio of....to.....*

PRINCIPLES: *1. The dividend, divisor and remainder must be similar numbers.*

2. The quotient must be abstract.

3. The dividend equals the product of the divisor and quotient, plus the remainder.

NOTE.—Many authors hold that the divisor may be abstract and the quotient similar to the dividend. Such a principle is inconsistent with the above definition of division. The author believes that the above definition is broad enough to cover every application of division. (See Art. 16.)

$$35 \div 5 = 7$$

is read *35 divided by 5 equals 7.*

$$35 : 5 = 7$$

is read *the ratio of 35 to 5 equals 7.* Each means that *35 things contain 5 of those things 7 times.* (See further explanation, Art. 17.) The *35* is the *dividend;* the *5*, the *divisor;* and the *7*, the *quotient.*

15. The Process of Short Division.—In *Short Division*, the process is performed mentally, and the dividend, divisor, and quotient only are written. Short division should be employed when the divisor does not exceed 12.

Put the divisor on the left of the dividend, and begin at the left of the dividend to divide. Write the quotient below.

It would be impossible to divide the whole of a large dividend at a glance; but the number represented by the first one or two figures on the left can be divided; the remainder, if any from this division, can be reduced to the next lower order and the division continued until all figures of the dividend are used. The following examples explain:

EXAMPLES.

1. Divide 76 by 4.

Process.

$$\begin{array}{r} 4\underline{)76} \\ 19 \end{array}$$

Explanation: (1) $7 \div 4 = 1$, and 3 remaining. Write the *1* below the 7.

(2) The remainder 3 is tens. 3 tens + 6 units = 36 units. $36 \div 4 = 9$. Write the *9* below. *Quotient 19.*

2. Divide 169 by 8.

Process.

$$\begin{array}{r} 8\underline{)169} \\ 21 \\ \text{rem. } 1 \end{array}$$

Explanation: (1) 1 of the dividend is smaller than 8, then we use 16.

(2) $16 \div 8 = 2$. Write the *2* below the *6*.

(3) $9 \div 8 = 1$, rem. 1. Write the quotient *1* to the right of the *2*, and the *rem. 1* below. *Quotient 21, remainder 1.*

3. Divide 7220 by 6.

Process.

$$\begin{array}{r} 6\underline{)7220} \\ 1203 \\ \text{rem. } 2 \end{array}$$

Explanation: (1) $7 \div 6 = 1$, rem. 1. Write the quotient *1* below the *7*.

(2) Remainder 1 is 1 thousand. 1 thousand + 2 hundred = 12 hundred. $12 \div 6 = 2$. Write the *2* below.

(3) $2 \div 6 = 0$, rem. 2. Write *0* below.

(4) The remainder 2 tens = 20 units. $20 \div 6 = 3$, rem. 2. Write the *3* at the right of the *0*, and the *rem. 2* below. *Quotient 1203, remainder 2.*

Tests of Accuracy: (1) *Review the work*, or (2) *apply principle 3.*

EXERCISE VIII

Copy and find quotients and remainders:

1.	*2.*	*3.*	*4.*	*5.*
8)328	6)372	9)405	7)595	5)895
6.	*7.*	*8.*	*9.*	*10.*
4)4567	6)8946	7)5938	3)9874	9)8064
11.	*12.*	*13.*	*14.*	*15.*
10)4892	11)5943	12)8736	11)9878	10)5306

Find quotients and remainders:

16. 84695÷11	*21.* 73024÷8	*26.* 235061÷12
17. 59060÷10	*22.* 16295÷11	*27.* 456789÷11
18. 73421÷12	*23.* 18020÷6	*28.* 760890÷9
19. 12345÷12	*24.* 16200÷12	*29.* 888888÷12
20. 23456÷9	*25.* 90000÷7	*30.* 987654÷8

16. Three Applications of Division.

$$9\overline{)36}$$
$$4$$

The fact of this division is, that 36 things contain 9 of those things *4 times.*

Note.—We must be careful not to fall into the error of thinking that 36 *contains* 9 as a bucket *contains* water. The word "*contains*," as here used, means "*is made up of,*" or "*is composed of.*"

First Application.—*Problem:* There are 36 apples in a lot. How many times can we take 9 apples from the lot?

> *Answer:* From division we know that *36 apples contain 9 apples 4 times.* Then, we can take 9 apples from the lot *4 times.*

This is the most *direct application* of division, and the only one which we have presented so far in this part.

SECOND APPLICATION.—*Problem:* There are 36 apples in one lot and 9 apples in another. The first lot is how many times as large as the second lot?

Answer: From division, we know that *36 apples contain 9 apples 4 times.* Then, 36 apples must be *4 times* as large as 9 apples of the same size.

This is the *application* nearest like the direct application, and may be called the **Ratio Idea of Division.**

THIRD APPLICATION.—*Problem:* There are 36 apples in a lot. What is $\frac{1}{9}$ of the lot?

Answer: From division, we know that *36 apples contain 9 apples 4 times.* Suppose we take 9 of the 36 apples, and with them begin 9 new lots; for every time we can take 9 apples from the 36 apples, we can put 1 apple in each new lot. But 36 apples contain 9 apples *4 times.* Therefore we can put 4 apples in each new lot. Thus, we have separated 36 apples into 9 equal new lots, and one lot contains $\frac{1}{9}$ of 36 apples, or 4 apples.

NOTE.—If necessary, the teacher should illustrate this with objects.

This may be called the **Fraction Idea of Division.**

17. Different Ways of Expressing Division. There are many ways or methods of expressing division. The following are in common use:

(1) $36 \div 9$

This expression is usually read, "*36 divided by 9.*" It means (1) 36 contains 9, (2) the ratio of 36 to 9, or (3) $\frac{1}{9}$ of 36. It is a general method of expressing division.

$$(2)\ \frac{36}{9}$$

This expression may be read, (1) "*36 divided by 9,*" (2) "*The ratio of 36 to 9,*" or (3) "*36 ninths.*" It, like the first expression, is a general method of expressing division.

$$(3)\ 36:9$$

Read, "*The ratio of 36 to 9.*" This expresses only the Ratio Idea of Division.

$$(4)\ \tfrac{1}{9} \text{ of } 36$$

Read, "*One-ninth of 36.*" This expresses only the Fraction Idea of Division.

18. Disposing of the Remainder.

EXAMPLE.

1. Divide 134 by 5.

$$5\overline{)134}$$
$$26\tfrac{4}{5}$$

Explanation: After we had obtained the quotient 26, we had a remainder of 4. This 4 is to be divided by 5. But we learned in Expression (2), Article 17, that 4 divided by 5 may be put in the form, $\frac{4}{5}$, and called "*four-fifths.*" This we put on the right of the *26* and call the expression, $26\frac{4}{5}$, "*twenty-six and four-fifths.*" $26\frac{4}{5}$ is the *exact quotient.*

When the exact quotient is an integer, the dividend is said to be *divisible* by the divisor. Thus,

$$36 \div 9 = 4.$$

36 is divisible by 9.

EXERCISE IX.

Find exact quotients:

1. $347 \div 7$	*6.* $834:7$	*11.* $\frac{1}{7}$ of 965	*16.* $\frac{238}{5}$
2. $545 \div 9$	*7.* $209:6$	*12.* $\frac{1}{9}$ of 677	*17.* $\frac{421}{12}$
3. $806 \div 11$	*8.* $707:11$	*13.* $\frac{1}{12}$ of 509	*18.* $\frac{856}{9}$
4. $793 \div 12$	*9.* $651:5$	*14.* $\frac{1}{10}$ of 327	*19.* $\frac{701}{6}$
5. $820 \div 9$	*10.* $939:10$	*15.* $\frac{1}{11}$ of 436	*20.* $\frac{409}{11}$

19. Long Division.—In *Long Division*, the process is the same as in short division, but the work is written. Write the quotient *above* or to the *right* of the dividend. The author believes that placing the quotient above has some advantages.

EXAMPLES.

1. Divide 464 by 17.

Process.

$$\begin{array}{r} 27 \\ 17\overline{)464} \\ \underline{34} \\ 124 \\ \underline{119} \\ 5 \text{ rem.} \end{array}$$

Explanation: (1) $46 \div 17 = 2+$.* $2 \times 17 = 34$. $46 - 34 = 12$. Bring down 4. $124 \div 17 = 7+$. $7 \times 17 = 119$. $124 - 119 = 5$. *Quotient 27, remainder 5.*

Since we must now deal with large divisors, it is not always easy to tell just what the quotient figure should be without trying. The following examples will show how to find the quotient figure by trial.

$$\begin{array}{r} 1\mathit{5} \\ 23\overline{)387} \\ \underline{23} \\ 157 \\ \underline{\mathit{115}} \\ \mathit{42} \end{array}$$

Explanation: Suppose we do not know how many times 23 is contained in 157, and we try *5 times.* $5 \times 23 = 115$. $157 - 115 = 42$. But 42 is larger than 23; therefore, 157 will contain 23 more than 5 times.

REMEMBER: *When the remainder is larger than the divisor the quotient figure is* **too small.**

$$\begin{array}{r} 1\mathit{7} \\ 23\overline{)387} \\ \underline{23} \\ 157 \\ \mathit{161} \end{array}$$

Explanation: If we do not know how many times 157 contains 23, suppose we try *7 times.* $7 \times 23 = 161$. But 161 is larger than 157; therefore, 157 will not contain 23 7 times.

REMEMBER: *When the number to be subtracted is larger than the number from which you are to subtract, the quotient figure is* **too large.**

---

* Read, 2+, "*two plus.*" It here means *2 and a remainder.*

2. Divide 101655 by 251.

Process.

```
      405
251)101655
    1004
      1255
      1255
```

Explanation: After bringing down the first *5* of the dividend, the 125 thus formed is too small to contain the divisor, 251. Put a *0* in the quotient, bring down the next figure, and proceed as before. *Quotient, 405.*

3. Divide 3400 by 200.

Process.

```
      17
2x00)34x00
```

Explanation: Cut off all 0's found on the right of the divisor, and an equal number of figures from the right of the dividend. Divide the remaining part of the dividend by the remaining part of the divisor. *Quotient, 17.*

4. Divide 3405 by 200.

Process.

```
      17 5/200
2x00)34x05
        5 rem.
```

Explanation: The 5 cut off is remainder. *Exact quotient,* $17\frac{5}{200}$.

5. Divide 3505 by 1700.

Process.

```
        2 105/1700
17x00)35x05
      34
       105 rem.
```

Explanation: If there be a remainder from the dividing, the part of the dividend cut off is annexed to form the complete remainder. *Exact quotient,* $2\frac{105}{1700}$.

6. Divide 34050 by 1000.

Process.

```
        34 50/1000
1x000)34x050
          50 rem.
```

Explanation: When the divisor is 10, 100, 1000, etc., the part cut off of the dividend is remainder, and the part not cut off is the integral quotient. *Exact quotient,* $34\frac{50}{1000}$.

EXERCISE X.

Find quotients:

1. 8547 ÷ 37	*7*. 45368 ÷ 106
2. 15170 ÷ 74	*8*. 111366 ÷ 207
3. 38916 ÷ 92	*9*. 176868 ÷ 306
4. 48418 ÷ 86	*10*. 258210 ÷ 342
5. 373520 ÷ 580	*11*. 150181 ÷ 179
6. 597760 ÷ 640	*12*. 177156 ÷ 259

13. 510788 ÷ 554
14. 324264 ÷ 458
15. 481866 ÷ 539
16. 363636 ÷ 468
17. 514152 ÷ 579
18. 316386 ÷ 378
19. 429586 ÷ 347
20. 355074 ÷ 249
21. 715464 ÷ 456
22. 800197 ÷ 673
23. 11304202 ÷ 1729
24. 100812054 ÷ 5649

Find integral quotients and remainders:

25. 8581 ÷ 231
26. 15317 ÷ 206
27. 39077 ÷ 370
28. 49092 ÷ 563
29. 374813 ÷ 644
30. 600181 ÷ 937
31. 45979 ÷ 433
32. 112559 ÷ 538
33. 176868 ÷ 578
34. 324264 ÷ 709

Find exact quotients:

35. 348657 ÷ 1000
36. 4897531 ÷ 10000
37. 7306477 ÷ 45000
38. 8907601 ÷ 10800
39. 325093 ÷ 72000
40. 98764379 ÷ 137000
41. 13951475 ÷ 958
42. 32657727 ÷ 75632
43. 7391555011 ÷ 8604
44. 26083952680 ÷ 36745

20. General Review.

EXERCISE XI.

1. Define addition, addend, sum.

2. Give the principles governing addition.

3. Using the following addends, explain *in full* the process of addition:

24750, 84723, 203079.

4. Mr. Jones has real estate worth $7560; personal property, such as household goods and stock, worth $1739; a stock of merchandise worth $8420; and a bank deposit of $3124. How much is Mr. Jones worth altogether?

5. The sum of 7084, 537, 9463, and one other number is 28221. Find the other number.

6. Explain fully the process of subtraction, using the following example:

$$3408-1736=(\quad)?$$

7. What is subtraction? Subtrahend? Minuend? Difference?

8. Give the principles governing subtraction.

9. A man worth $8750 lost by fire a house worth $1280. What was he then worth?

10. The remainder is 756, and the subtrahend, 3146. Find the minuend.

11. The minuend and remainder are 14255 and 6728 respectively. Find the subtrahend.

12. Explain fully the process of multiplication, using the following example:

$$72\times 8407=(\quad)?$$

13. Define multiplication, multiplicand, multiplier, product.

14. Give the principles governing multiplication.

15. $346\times(\quad)=195144$?

16. $(\quad)\times 730=88330$?

17. A has 246 cattle; but B has 26 times as many as A. How many cattle has B?

18. Two numbers multiplied together give a product of 463686; one of the numbers is 654. Find the other number.

19. Define division, divisor, dividend, quotient, remainder, short division, long division.

20. Give the principles governing division.

21. Explain fully the process of short division, using the following example:

$$758646\div 9=(\quad)?$$

22. Explain fully the process of long division, using the following example:

$$195144\div 564=(\quad)?$$

23. By what rule do you know when the quotient figure is too large?

24. By what rule do you know when the quotient figure is too small?

25. How many \$1000-shares in a capital stock of \$2000000?

26. $194733 \div (\quad) = 231$?

27. $(\quad) \div 1248 = 321$?

28. $239761 \div (\quad) = 375$, remainder 136?

29. $74475 \div 324 = 229$, remainder $(\quad)$?

30. How do you shorten the process of multiplication when there are 0's on the right of the multiplier or multiplicand?

31. How do you shorten the process of division when there are 0's on the right of the divisor?

32. A and B start from the same place: A goes east 560 miles; B goes west 489 miles. Find the distance between them.

33. A and B start from the same place and travel in the same direction—A, 1846 rods; B, 359 rods. Find the distance between them.

34. What is the number, from which, if 734 be taken, the remainder is 591?

35. What is the number, to which, if 7428 be added, the sum will be 24173?

36. Find the number, from which, if 430 be subtracted, the remainder divided by 79, the quotient will be 24.

37. What number is that, from which, if 765 be subtracted, the remainder multiplied by 134, and 1795 be added to the product, the sum will be 233347?

38. If 375 be subtracted from the difference between two numbers, 126 will remain. 395 is the smaller number; find the larger.

39. Four men have \$7500: the 1st has \$1500; the 2d, \$1284; the 3d, \$2179. How much has the 4th?

40. Find the sum of four numbers, if the 1st is 177, the 2d is 316 more than the 1st, the 3d is 741 more than the 2d, and the 4th is 658 less than the 3d.

G. SIMPLIFYING NUMERICAL EXPRESSIONS.

21. A Term.—When a *numerical expression* is separated into parts by either of the signs, plus (+) or minus (−), these parts are called **Terms**; when not so separated, the expression itself is a *term*. Thus,

$$5-2+7=10$$

5, 2, 7, and 10 are terms.

Sometimes a term is composed of two or more numbers joined together by signs, × or ÷. Thus,

$$3\times4-8\div2$$

3×4, and $8\div2$, are terms.

Two or more terms may be inclosed by parentheses, ( ), and considered as a single term. Thus,

$$(4+5)-(10-4+2)$$

$(4+5)$ and $(10-4+2)$ are terms.

When an expression in parentheses is to be multiplied by a number, that number is usually placed just before the parentheses. Thus,

$$5(4+5)=45$$

$5(4+5)$ means *5 times the sum of 4 and 5*, or 5×9.

A dividend or a divisor may be composed of two or more numbers joined by signs. Thus,

$$\frac{5(4+5)}{12-3}=5(4+5)\div(12-3).$$

These two expressions are equal as indicated. The dividend is 5 times the sum of 4 and 5, and the divisor is the difference between 12 and 3. Each expression is a *term*.

22. Process.—In simplifying a numerical expression, (1) always simplify each term by itself and (2) add or subtract the terms as indicated by the sign, plus or minus.

EXAMPLES.

1. Simplify $7-(9-5)+4\times5-20\div4=(\quad)$?

$$(1)\ 7-(9-5)+4\times5-20\div4.$$
$$(2)\ 7-4+20-5=18, \text{ result.}$$

Explanation: 1st term, 7; 2d term, $9-5$ or 4; 3d term, 4×5 or 20; 4th term, $20\div4$ or 5.

2. Simplify $5(12-10)+4\times6-(2+9-3)$.

$$(1)\ 5(12-10)+4\times6-(2+9-3).$$
$$(2)\ 10+24-8=26, \text{ result.}$$

3. $\frac{8(9-4)}{10}-27\div9+3(5+3-7)=(\quad)$?

$$(1)\ \frac{8(9-4)}{10}-27\div9+3(5+3-7).$$
$$(2)\ 4-3+3=4, \text{ result.}$$

EXERCISE XII.

Simplify:

1. $4-3+10-5+12$

2. $12\div4+7-6+14$

3. $24-6\times3+4\times5\times3$

4. $74-10\times7+8\times3$

5. $8-(4-2)+(9-3)-(8-5)$

6. $9+(7+1)+2(10-4)+5\times6$

7. $\frac{8+7}{10-5}+\frac{21-3}{4+2}-\frac{2(7+3)}{9-5}$

8. $5\times8-\frac{3(5+3)}{12}+(25+5)\div(11-5)$

9. $\frac{34-20}{7}+\frac{72}{3(7-4)}-7+8\times5-7(12-8+4-2)$

10. $\frac{5(8+2)}{2(7-2)}+64\div(3+5)-(5+11-4)$

H. DERIVED OPERATIONS.

1. FACTORING.

23. Definitions and Principles.—A **Prime Number** is an integer that cannot be formed by multiplying two or more other integers together. Below is a table of prime numbers, to be used for reference:

TABLE OF PRIME NUMBERS FROM 1 TO 1000.

1	59	139	233	337	439	557	653	769	883
2	61	149	239	347	443	563	659	773	887
3	67	151	241	349	449	569	661	787	907
5	71	157	251	353	457	571	673	797	911
7	73	163	257	359	461	577	677	809	919
11	79	167	263	367	463	587	683	811	929
13	83	173	269	373	467	593	691	821	937
17	89	179	271	379	479	599	701	823	941
19	97	181	277	383	487	601	709	827	947
23	101	191	281	389	491	607	719	829	953
29	103	193	283	397	499	613	727	839	967
31	107	197	293	401	503	617	733	853	971
37	109	199	307	409	509	619	739	857	977
41	113	211	311	419	521	631	743	859	983
43	127	223	313	421	523	641	751	863	991
47	131	227	317	431	541	643	757	877	997
53	137	229	331	433	547	647	761	881	

A **Composite Number** is an integer that can be formed by multiplying two or more other integers together. As,

4, 6, 8, 9, 10, 12, 15, 16.

An **Even Number** ends in 0, 2, 4, 6, or 8; all others are **Odd Numbers.**

Factors of a number are those numbers which, multiplied together, form that number. As,

$$3 \times 5 \times 7 = 105.$$

3, 5, and 7 are factors of 105.

A **Prime Factor** is a prime number that is a factor.

A **Composite Factor** is a composite number that is a factor. Thus,

$$3 \times 4 = 12.$$

3 is a prime factor of 12; 4 is a composite factor of 12.

A **Common Factor** of two or more numbers is a factor of each of them. Thus,

$$7 \times 2 = 14; \quad 7 \times 3 = 21.$$

7 is a common factor of 14 and 21.

Two numbers that have no common integral factor (except one) are *prime to each other*. Thus,

$$3 \times 7 = 21$$
$$5 \times 11 = 55$$

21 and 55 are prime to each other.

A number can have but one set of prime factors, but a number that has three or more prime factors (besides one) may have more than one set of factors, some of which are composite. Thus,

$$30 = 3 \times 2 \times 5;$$

but $$30 = 3 \times 10 = 2 \times 15 = 6 \times 5.$$

30 has but one set of prime factors, 2, 3, 5; but it has three sets of factors, some of which are composite, 3, 10; 2, 15; 6, 5.

A **Multiple** of a number is a number that contains that number an integral number of times. Thus,

$$24 \div 12 = 2.$$

24 is a multiple of 12.

A **Common Multiple** of two or more numbers is a number that is a multiple of each of them. Thus,

$$24 \div 6 = 4. \qquad 24 \div 8 = 3.$$

24 is a common multiple of 6 and 8.

A **Divisor** of a number is a number that will divide that number and give an exact integral quotient. Thus,

$$24 \div 4 = 6.$$

4 is a divisor of 24.

A **Common Divisor** of two or more numbers is a number that is a divisor of each of them. Thus,

$$24 \div 6 = 4.$$
$$42 \div 6 = 7.$$

6 is a common divisor of 24 and 42.

GENERAL PRINCIPLES: *1. The multiplier and multiplicand are factors of the product.*

2. The divisor and the exact quotient are factors of the dividend.

3. Any composite number is equal to the product of all its prime factors.

4. All of any set of factors of a concrete number must be abstract, except one factor, which must be concrete and similar to the number itself.

NOTE.—It is immaterial which factor is considered concrete. For example, factor the number 30 bushels.

$2 \times 3 \times 5$ bushels = 30 bushels.
$2 \times 5 \times 3$ bushels = 30 bushels.
$5 \times 3 \times 2$ bushels = 30 bushels.
$2 \times 5 \times 3 \times 1$ bushel = 30 bushels.

5. A factor of a number is a factor of any multiple of that number.

6. A factor of any two numbers is also a factor of their sum and their difference.

7. When a number is divided by one of its prime factors, the quotient is the product of all the remaining prime factors of the number.

PRINCIPLES TO BE USED IN INSPECTING A NUMBER FOR FACTORS: *1. One factor of a number is 2, if the number ends in 0, 2, 4, 6, or 8.*

2. One factor of a number is 3, if 3 is a factor of the sum obtained by adding the digits of the number.

3. One factor of a number is 4, if 4 is a factor of the number expressed by the two right-hand figures.

4. One factor of a number is 5, if the number ends in 0 or 5.

5. One factor of a number is 6, if 2 and 3 are factors. (See principles 1 and 2.)

6. One factor of a number is 7, if 7 is a factor of the sum of once the first figure, plus 3 times the second, plus 2 times the third, plus 6 times the fourth, plus 4 times the fifth, plus 5 times the sixth, plus once the seventh, plus 3 times the eighth, and so on, following the order 1, 3, 2, 6, 4, 5 times the successive figures. Thus,

7 is a factor of 34755.

For, by the principle,

$$1\times5+3\times5+2\times7+6\times4+4\times3=70,$$

and 7 is a factor of 70.

7. One factor of a number is 8, if 8 is a factor of the number expressed by the three right-hand figures.

8. One factor of a number is 9, if 9 is a factor of the sum obtained by adding its digits.

9. One factor of a number is 10, if the number ends in 0.

NOTE.—The 1st, 2d and 4th of these principles are most practical. The 3d, 8th and 9th are often used. The 5th and 7th are sometimes used. But the 6th is not used in practice; for it takes longer to apply the principle than to divide by 7.

24. Factoring by Inspection.—When numbers are small, they are usually factored by inspection.

EXAMPLES.

1. Find the factors of 9.

Think: $9=3\times3$. *Call:* "Factors 3, 3."

2. Factor 160.

Think: $160=2\times2\times2\times2\times2\times5$. *Call:* "Factors, 2, 2, 2, 2, 2, 5."

EXERCISE XIII.

Factor by inspection :

1. 25	*4.* 72	*7.* 112	*10.* 210
2. 40	*5.* 96	*8.* 128	*11.* 175
3. 75	*6.* 91	*9.* 150	*12.* 240

25. Factoring by Division.—When the number to be factored is large, *division* is employed in finding the factors. (See General Principles 2 and 7.) The successive divisors are (1) found by inspection (see Principles to be used in Inspecting) or (2) by trial.

EXAMPLES.

1. Find the prime factors of 210.

Process.

$$\begin{array}{r|r} 5 & 210 \\ \hline 3 & 42 \\ \hline 2 & 14 \\ \hline & 7 \end{array}$$

Explanation: (1) By Prin. 4, 5 is a factor of 210.

(2) By Prin. 2, 3 is a factor of 42.

(3) By Prin. 1, 2 is a factor of 14.

∴ 5, 3, 2, and 7 are the factors of 210.

2. Find the prime factors of 5040. Find 5 composite factors.

Process.

(1)
$$\begin{array}{r|r} 2 & 5040 \\ \hline 2 & 2520 \\ \hline 2 & 1260 \\ \hline 2 & 630 \\ \hline 5 & 315 \\ \hline 3 & 63 \\ \hline 3 & 21 \\ \hline & 7 \end{array}$$

Factors, 2, 2, 2, 2, 5, 3, 3, 7.

(2) $2\times2=4$; $2\times2\times2=8$; $2\times3=6$; $2\times2\times3=12$; $2\times7=14$.

NOTE.—Two or more prime factors of a number multiplied together always give a composite factor of that number.

EXERCISE XIV.

Find the prime factors of—

1. 2304	*4.* 17160	*7.* 73920	*10.* 282960
2. 2835	*5.* 20304	*8.* 39375	*11.* 146146
3. 3000	*6.* 32340	*9.* 36288	*12.* 1067220

Find fourteen factors for each of the following:

13. 210	*15.* 1155	*17.* 770
14. 330	*16.* 462	*18.* 510

26. Factoring Expressions Composed of Two or More Terms.—It is sometimes necessary or convenient to factor an expression consisting of two or more terms. This may be done by dividing by a factor common to all the terms.

EXAMPLES.

1. Take the factor 5 out of $20+25$.

Process: $20+25=5(4+5)$.

Explanation: 5 is contained in 20 4 times, and in 25 5 times. Thus, 5 is one factor and $4+5$ is the other. $5(4+5)$ may not only be interpreted as *5 times the sum of 4 and 5*, but also *5 times 4 added to 5 times 5.*

2. Take the factor 5 out of $25-20$.

Process: $25-20=5(5-4)$.

Explanation: $5(5-4)$ may be interpreted *5 times the difference between 5 and 4*, or *5 times 5 minus 5 times 4.*

3. Take the common factor out of $49-14+28$.

Process: $49-14+28=7(7-2+4)$.

EXERCISE XV.

Take out the common factors:

1. 24−4	*6.* 5+20−15
2. 72+18	*7.* 18−9+36
3. 36−16	*8.* 48−16+64
4. 57+38	*9.* 25+30−45
5. 75−50	*10.* 14+28−21+7

27. Principles of Multiplication and Division Relating to Factors.— Factors are of common occurrence in the processes of multiplication and division, and the following principles will be of service:

PRINCIPLES: *1. Multiplying any factor of a product by a number, multiplies the product by that number.*

Example: Indicate the multiplication of $3\times5\times4$ by 2. (3 forms.)

Forms: (1) $(2\times3)\times5\times4=6\times5\times4$; or,
(2) $3\times(2\times5)\times4=3\times10\times4$; or,
(3) $3\times5\times(2\times6)=3\times5\times12$.

2. Dividing any factor of a product by a number, divides the product by that number.

Example: Indicate the division of $4\times6\times8$ by 2. (3 forms.)

Forms: (1) $(4\div2)\times6\times8=2\times6\times8$; or,
(2) $4\times(6\div2)\times8=4\times3\times8$; or,
(3) $4\times6\times(8\div2)=4\times6\times4$.

3. A quotient is multiplied by a number (1) *by multiplying its dividend by that number, or* (2) *by dividing its divisor by that number.*

Example: Indicate the multiplication of $24\div6$ by 3. (2 forms.)

Forms: (1) $(3\times24)\div6=72\div6$; or,
(2) $24\div(6\div3)=24\div2$.

NOTE.—The dividend or divisor may be expressed as a product of two or more factors.

Example: Indicate the multiplication of $\frac{12\times5}{9\times6}$ by 3. (4 forms.)

Forms: (1) $\frac{(3\times12)\times5}{9\times6}=\frac{36\times5}{9\times6}$; or,

(2) $\frac{12\times(3\times5)}{9\times6}=\frac{12\times15}{9\times6}$; or,

(3) $\frac{12\times5}{(9\div3)\times6}=\frac{12\times5}{3\times6}$; or,

(4) $\frac{12\times5}{9\times(6\div3)}=\frac{12\times5}{9\times2}$.

4. A quotient is divided by a number (1) by dividing its dividend by that number, or (2) by multiplying its divisor by that number.

Example 1: Indicate the division of $\frac{12}{2}$ by 3. (2 forms.)

Forms: (1) $\frac{12\div3}{2}=\frac{4}{2}$; or,

(2) $\frac{12}{3\times2}=\frac{12}{6}$.

Example 2: Indicate the division of $\frac{12\times18}{4\times11}$ by 3. (4 forms.)

Forms: (1) $\frac{(12\div3)\times18}{4\times11}=\frac{4\times18}{4\times11}$; or,

(2) $\frac{12\times(18\div3)}{4\times11}=\frac{12\times6}{4\times11}$; or,

(3) $\frac{12\times18}{(3\times4)\times11}=\frac{12\times18}{12\times11}$; or,

(4) $\frac{12\times18}{4\times(3\times11)}=\frac{12\times18}{4\times33}$.

EXERCISE XVI.

Indicate as above the multiplication of—

1. 4×7 by 8. (2 forms.)
2. $9\times12\times3$ by 5. (3 forms.)
3. $\frac{150}{75}$ by 25. (2 forms.)
4. $\frac{147}{14\times21}$ by 7. (3 forms.)
5. $\frac{6\times5}{10}$ by 5. (3 forms.)
6. $\frac{14\times24}{9\times18\times3}$ by 3. (5 forms.)

Indicate as above the division of—

7. 12×30 by 6. (2 forms.)
8. $15\times20\times45$ by 5. (3 forms.)
9. $\frac{144}{9}$ by 12. (2 forms.)
10. $\frac{1250}{5\times10}$ by 25. (3 forms.)
11. $\frac{120\times360}{9}$ by 12. (3 forms.)
12. $\frac{42\times84}{25\times6}$ by 7. (4 forms.)

2. CANCELLATION.

28. Definition and Principle.—Cancellation is a process of shortening the work of division by omitting the same factors from both dividend and divisor.

PRINCIPLE: *Dividing both dividend and divisor by the same number does not affect the quotient.*

NOTE.—Dividing the dividend, *divides* the quotient (Art. 27, Prin. 4). Dividing the divisor, *multiplies* the quotient (Art. 27, Prin. 3). But, when the quotient is both multiplied and divided by the same number, its value is unchanged.

29. Process.—It is usual to write the dividend or the factors of the dividend above a horizontal line, the divisor or factors of the divisor below. Proceed to cancel and reject factors common to both dividend and divisor. When all common factors are rejected, divide the product of the remaining factors of the dividend by the product of the remaining factors of the divisor.

EXAMPLES.

1. Divide 50×96 by 15×8.

$$\textit{Process:}\ \frac{\overset{10}{\cancel{50}}\times\overset{\overset{4}{\cancel{12}}}{\cancel{96}}}{\underset{\cancel{3}}{\cancel{15}}\times 8}=40.$$

Explanation: (1) Cancel 5 out of 50 and 15, leaving 10 above and 3 below. (2) 96 contains 8 12 times; cancel 8 and 96, placing 12 above. (3) 12 contains 3 4 times; cancel 3 and 12, and place 4 above. (4) $4\times10=40$.

2. How many times will $48\times27\times15$ contain $9\times6\times10$?

$$\textit{Process:}\ \frac{\overset{\overset{4}{\cancel{8}}}{\cancel{48}}\times\overset{3}{\cancel{27}}\times\overset{3}{\cancel{15}}}{\cancel{9}\times\cancel{6}\times\underset{\cancel{2}}{\cancel{10}}}=36.$$

Explanation: (1) 6 in 48 8 times. Cancel 6 and 48, and place 8 above. (2) 9 in 27 3 times. Cancel, and write 3

above. (3) 5 is common to 15 and 10. Cancel, and write 3 above and 2 below. (4) 2 in 8 4 times. Cancel, and write 4 above. (5) $4\times3\times3=36$.

EXERCISE XVII.

1. $\dfrac{17\times13\times50}{65\times10}=(\ )?$

2. $\dfrac{72\times105\times160}{24\times42\times120}=(\ )?$

3. $\dfrac{200\times57\times35}{75\times70\times19}=(\ )?$

4. $\dfrac{441\times210\times216}{126\times105\times84}=(\ )?$

5. $\dfrac{625\times720\times81}{750\times729}=(\ )?$

6. $\dfrac{400\times375\times338}{507\times625\times80}=(\ )?$

7. $\dfrac{2816\times16200\times8400}{528\times1512\times1280}=(\ )?$

8. $\dfrac{5409\times1728\times840}{8640\times10217}=(\ )?$

3. GREATEST COMMON DIVISOR.

30. Definition and Principles.—The **Greatest Common Divisor** of two or more numbers is the largest number that is contained in each of them an integral number of times.

PRINCIPLES: *1. Every factor of a number is a divisor of that number.*

2. The product of two or more prime factors of a number is a divisor of that number.

3. A divisor of a number is a divisor of any multiple of that number.

4. A common divisor of two numbers is a divisor of their sum and of their difference.

5. The G. C. D. of two or more numbers is the product of all the prime factors common to the numbers.

6. The G. C. D. of two numbers is the G. C. D. of either of them and their sum, or difference.

31. Process.—When the numbers are small, the G. C. D. is usually found by factoring.

EXAMPLES.

1. Find the G. C. D. of 24, 48, and 72.

Process.

$24 = 2 \times 2 \times 2 \times 3.$
$48 = 2 \times 2 \times 2 \times 2 \times 3.$
$72 = 2 \times 2 \times 2 \times 3 \times 3.$
2, 2, 2, 3 are common factors.
$\therefore 2 \times 2 \times 2 \times 3 = 24$, G. C. D.

The following process is also used when the numbers are small:

2. Find the G. C. D. of 15, 45, and 60

Process.

```
5)15, 45, 60
3) 3,  9, 12
   1,  3,  4
```

3 and 5 are all the common factors.
$\therefore 3 \times 5 = 15$, G. C. D.

NOTE.—In this process, use only such divisors as will divide all the numbers.

When the numbers are so large as not to be easily factored by inspection, the following process is used:

3. Find the G. C. D. of 254 and 381.

Process.

```
       1
254)381
    254     2
    127)254
        254
```

Explanation: Since 127 is the G. C. D. of 127 and 254, it is also the G. C. D. of 254 and 381 (Prin. 6). $\therefore$ the required G. C. D. is 127.

NOTE.—This plan or process of finding the G. C. D. is to divide the larger number by the smaller; then divide the divisor by the remainder; then continue to divide the last divisor by the last remainder, until there is no remainder. The last divisor is the required G. C. D.

4. Find the G. C. D. of 221, 364, and 5512.

Process.

```
(1)                    (2)
     15                     4
364)5512               52)221
    364                   208  4
    1872                     13)52
    1820    7                    52
        52)364
           364
```

Explanation: (1) The G. C. D. of 364 and 5512 is 52. But 52 is not a factor of 221. (2) The G. C. D. of 52 and 221 is 13. Now, 52 contains all factors common to 364 and 5512; 13 is the only factor common to 52 and 221. Therefore 13 is the G. C. D. of 221, 364, and 5512.

∴ the required G. C. D. is 13.

EXERCISE XVIII.

Find the G. C. D. of—

1. 36, 48, 72.
2. 96, 120, 1728.
3. 210, 840, 126.
4. 1022, 2513.
5. 1573, 6331.
6. 1980, 2511.
7. 4340, 12504.
8. 3070, 2149, 614.
9. 7384, 6495, 4321.
10. 1360, 1632, 2040, 4080.

4. LEAST COMMON MULTIPLE.

32. Definition and Principles.—The **Least Common Multiple** of two or more numbers is the smallest number that contains each of them an integral number of times.

PRINCIPLES: *1. A multiple of a number is divisible by that number; and a common multiple of two or more numbers is divisible by each of them.*

2. A multiple of a number contains all the prime factors of that number; and a common multiple of two or more numbers contains all the prime factors of each of them.

3. The L. C. M. of two or more numbers contains every prime factor of the several numbers the greatest number of times it is found in any one of them.

4. The L. C. M. of two or more numbers is a multiple of all the factors of those numbers.

5. The L. C. M. is the product of one of the numbers multiplied by the quotient obtained by dividing the other numbers by their G. C. D.

NOTE.—The L. C. M. of more than two numbers may be found by finding the L. C. M. of two of them; then, of that result and the third; then, of that result and the fourth, and so on.

33. Process.—The L. C. M. of two or more numbers may be easily found by factoring, when the numbers are small.

EXAMPLES.

1. Find the L. C. M. of 15, 12, and 10.

Process.

$15 = 3 \times 5$.
$12 = 3 \times 2 \times 2$.
$10 = 5 \times 2$.
The L. C. M. must contain 3 once, 5 once, and 2 twice. (Prin. 3.)
$\therefore 3 \times 5 \times 2 \times 2 = 60$, L. C. M.

Division is often employed in finding the L. C. M.

2. Find the L. C. M. of 10, 12, 24, and 30.

Process.

$$\begin{array}{r|llll} 2 & 10, & 12, & 24, & 30 \\ \hline 2 & 5, & 6, & 12, & 15 \\ \hline 5 & 5, & 3, & 6, & 15 \\ \hline 3 & 1, & 3, & 6, & 3 \\ \hline & 1, & 1, & 2, & 1 \end{array}$$

$2 \times 2 \times 5 \times 3 \times 2 = 120$, L. C. M.

NOTE.—In this process, divide by any *prime* number that will divide two or more of the numbers. Numbers not containing the divisor an integral number of times must be brought down; as 5 and 15 in the second division above. When the final results are prime to each other, the continued product of all the *divisors* and final results (1's may be omitted) will be the L. C. M. (Prin. 3.)

When some of the numbers are factors of others, these factors may be omitted in finding the L. C. M. (Prin. 4.)

3. Find the L. C. M. of 21, 15, and 45.

Process.

$3\,)\,21,\ \cancel{15},\ 45$

$\quad 7, \quad 15$

$3 \times 7 \times 15 = 315$, L. C. M.

Explanation: The 15 may be canceled from the process, for it is a factor of 45.

The L. C. M. of two numbers may be found by principle 5, as follows:

4. Find the L. C. M. of 160, 240, and 230.

I. Find the L. C. M. of 160 and 240.

(1)

```
       1
160)240
    160    2
     80)160
        160
```

∴ 80 is G. C. D. of 160 and 240.

(2) $160 \div 80 = 2$.

(3) $2 \times 240 = 480$, the L. C. M. of 160 and 240.

II. Find the L. C. M. of 480 and 230.

(4)

```
       2
230)480
    460   11
     20)230
        20
        30
        20   2
        10)20
           20
```

∴ 10 is G. C. D. of 480 and 230.

(5) $230 \div 10 = 23$.

(6) $23 \times 480 = 11040$, the required L. C. M.

EXERCISE XIX.

Find the L. C. M. of—

1. 42, 50, 72.
2. 120, 96, 48.
3. 144, 240, 600.
4. 256, 120, 360.
5. 81, 540, 360.
6. 24, 36, 48, 60.
7. 75, 125, 50, 100.
8. 105, 84, 63, 147.
9. 174, 485, 4611, 970.
10. 264, 144, 324, 576.
11. 1260, 198, 480, 330.
12. 2862, 3498, 4158.

I. COMMON FRACTIONS.

34. Definitions and Principles.—A **Fraction** is a number composed of *fractional units.*

What is a unit? Integral unit? Fractional unit?

A **Common Fraction** is a fraction expressed by two numbers, placed one above and the other below a horizontal line. Thus,

$$\frac{5}{17} \qquad \frac{346}{9}$$

The number below the line is called the **Denominator,** and tells the size of the fractional unit. The number above the line is called the **Numerator,** and tells the number of units in the fraction. The numerator and denominator are called the **Terms** of the fraction.

5 ....Numerator.
17..Denominator.

Explanation: The "*17*" tells that the units are *seventeenths;* the "*5*" tells that this fraction expresses *five* units—*five-seventeenths.*

A *common fraction* is also an expression of division. Review Article 17.

The *numerator* of the fraction is the *dividend*, and the *denominator*, the *divisor.*

When the numerator is equal to or larger than the denominator, the fraction is called an **Improper Fraction.**

PRINCIPLES: *1. A fraction is multiplied* (1) *by multiplying its numerator,* or (2) *by dividing its denominator.* (See Prin. 3, Art. 27.)

ILLUSTRATION.

Multiply $\frac{1}{6}$ by 3.

Process: (1) $3 \times \frac{1}{6} = \frac{3 \times 1}{6} = \frac{3}{6}$, or,

(2) $3 \times \frac{1}{6} = \frac{1}{6 \div 3} = \frac{1}{2}$.

The results agree; for $\frac{3}{6}$ of a number $= \frac{1}{2}$ of it.

2. A fraction is divided (1) *by dividing its numerator*, or (2) *by multiplying its denominator.* (See Prin. 4, Art. 27.)

ILLUSTRATION.

Divide $\frac{2}{3}$ by 2.

Process: (1) $\frac{2}{3} \div 2 = \frac{2 \div 2}{3} = \frac{1}{3}$; or,

(2) $\frac{2}{3} \div 2 = \frac{2}{2 \times 3} = \frac{2}{6}$.

These results agree; for $\frac{1}{3}$ of a number $= \frac{2}{6}$ of it.

3. The value of a fraction is not changed (1) *if both its terms are multiplied by the same number*, or (2) *if both its terms are divided by the same number.*

ILLUSTRATIONS.

1. Multiply both terms of $\frac{1}{2}$ by 3.

Process: $\frac{3 \times 1}{3 \times 2} = \frac{3}{6}$.

But $\frac{3}{6} = \frac{1}{2}$. Therefore, the value is not changed.

2. Divide both terms of $\frac{4}{6}$ by 2.

Process: $\frac{4 \div 2}{6 \div 2} = \frac{2}{3}$.

But $\frac{2}{3} = \frac{4}{6}$. Therefore, the value is not changed.

35. Reducing Fractions to Higher Terms.—A fraction is reduced to higher terms by multiplying both its terms by the same number, according to principle 3. This does not change its value.

EXAMPLES.

1. Reduce $\frac{2}{3}$ to 9ths.

Process: $\frac{2}{3} = \frac{3 \times 2}{3 \times 3} = \frac{6}{9}$, result.

Why multiply by 3? Because the denominator 3 is to be changed to 9. $3 \times 3 = 9$.

NOTE.—An integer may be treated as a fraction, having 1 for a denominator. $\frac{5}{1} = 5$ or $5 = \frac{5}{1}$. $\frac{5}{1}$ is read *5 ones.*

2. Reduce 7 to 5ths.

Process: $\frac{7}{1}=\frac{5\times7}{5\times1}=\frac{35}{5}$, result.

3. Reduce $\frac{5}{7}$ to 63ds.

Process: (1) $63\div7=9$.

(2) $\frac{5}{7}=\frac{9\times5}{9\times7}=\frac{45}{63}$, result.

Why divide 63 by 7?

4. Reduce $\frac{7}{13}$ to 195ths.

Process:

$$\begin{array}{r} 15 \\ 13\overline{)195} \\ \underline{13} \\ 65 \\ \underline{65} \end{array}$$

(1) as above. (2) $\frac{7}{13}=\frac{15\times7}{15\times13}=\frac{105}{195}$, result.

EXERCISE XX.

Reduce—

1. $\frac{5}{12}$ to 48ths.

2. $\frac{8}{21}$ to 63ds.

3. 11 to 37ths.

4. $3\frac{5}{19}$ to 209ths.

5. $\frac{12}{30}$ to 390ths.

6. $\frac{11}{27}$ to 324ths.

7. $\frac{15}{37}$ to 333ds.

8. $9\frac{4}{19}$ to 2109ths.

9. $\frac{17}{59}$ to 708ths.

10. $18\frac{75}{101}$ to 8181sts.

36. Reducing Fractions to Lowest Terms.—

A fraction is in its *lowest terms*, when its numerator and denominator are prime to each other. Divide both numerator and denominator by their G. C. D., or reject all factors common to both numerator and denominator. This does not change its value.

EXAMPLES.

1. Reduce $\frac{4}{6}$ to thirds.

Process: $\frac{4}{6}=\frac{4\div2}{6\div2}=\frac{2}{3}$, result.

Why divide by 2? How many 3's in 6?

2. Reduce $\frac{12}{18}$ to lowest terms.

Divide by G. C. D. of both terms. The G. C. D. of 12 and 18 is 6.

Process: $\frac{12}{18}=\frac{12\div6}{18\div6}=\frac{2}{3}$, result; or $\frac{12}{18}=\frac{6)12}{6)18}=\frac{2}{3}$, result.

3. Reduce $\frac{49}{161}$ to lowest terms.

G. C. D. of 49 and 161 = 7.

Process: $\frac{49}{161}=\frac{49\div7}{161\div7}=\frac{7}{23}$, result; or $\frac{49}{161}=\frac{7)\,49}{7)\,161}=\frac{7}{23}$, result.

EXERCISE XXI.

Reduce to lowest terms—

1. $\frac{96}{360}$	*4.* $\frac{35}{140}$	*7.* $\frac{889}{127}$	*10.* $\frac{156}{663}$
2. $\frac{162}{342}$	*5.* $\frac{244}{444}$	*8.* $\frac{1000}{3600}$	*11.* $\frac{1767}{4557}$
3. $\frac{48}{135}$	*6.* $\frac{1430}{55}$	*9.* $\frac{308}{630}$	*12.* $\frac{19404}{37044}$

37. Reducing Mixed Numbers to Improper Fractions.

—Reduce the integral part to the denomination of the fractional part and add. The value is not changed.

EXAMPLES.

1. Reduce $3\frac{1}{2}$ to halves.

Process: (1) $3=\frac{2\times3}{2\times1}=\frac{6}{2}$. (2) $\frac{6}{2}+\frac{1}{2}=\frac{7}{2}$, result.

(1) Reduce the integer to halves, (2) add the $\frac{1}{2}$.

Shorter Form: $3\frac{1}{2}=\frac{2\times3}{2}+\frac{1}{2}=\frac{7}{2}$, result.

2. $7\frac{3}{4}$ to 4ths.

Process: $7\frac{3}{4}=\frac{4\times7}{4}+\frac{3}{4}=\frac{31}{4}$, result.

3. Reduce $17\frac{9}{23}$ to an improper fraction.

Mechanical Work:

$$\begin{array}{r}
17 \\
\underline{23} \\
51 \\
\underline{34} \\
391 \\
\underline{9} \\
400
\end{array}$$

Explanation: 17 reduced to twenty-thirds will make 23×17, or 391 twenty-thirds. Adding the 9 twenty-thirds, we have 400 twenty-thirds, or $\frac{400}{23}$.

Result, $\frac{400}{23}$.

EXERCISE XXII.

Reduce to improper fractions:

1. $17\frac{3}{11}$	*4.* $18\frac{1}{18}$	*7.* $98\frac{11}{27}$	*10.* $371\frac{8}{29}$
2. $24\frac{4}{9}$	*5.* $75\frac{5}{17}$	*8.* $304\frac{5}{12}$	*11.* $488\frac{111}{1111}$
3. $13\frac{9}{25}$	*6.* $89\frac{17}{24}$	*9.* $132\frac{15}{16}$	*12.* $77\frac{587}{2347}$

38. Reducing Improper Fractions to Integers or Mixed Numbers.

—This is done by division. See Article 17. The value is not changed.

EXAMPLES.

1. Reduce $\frac{20}{4}$ to an integer.

Process:

$$\begin{array}{l} 4\underline{)20} \\ \quad 5, \text{ result.} \end{array}$$

Note.—As an expression of division, "*20*" is the dividend and "*4*" the divisor.

2. Reduce $\frac{355}{18}$ to a mixed number.

Process:

$$\begin{array}{r} 19\frac{13}{18}, \text{ result.} \\ 18\overline{)355} \\ \underline{18} \\ 175 \\ \underline{162} \\ 13 \end{array}$$

Note.—If the fractional part of the result is not in its lowest terms, reduce by Article 36.

EXERCISE XXIII.

Reduce to an integer or mixed number:

1. $\frac{132}{12}$	*4.* $\frac{1284}{95}$	*7.* $\frac{2121}{201}$	*10.* $\frac{4583}{125}$
2. $\frac{596}{27}$	*5.* $\frac{1895}{342}$	*8.* $\frac{3600}{400}$	*11.* $\frac{9374}{330}$
3. $\frac{843}{72}$	*6.* $\frac{1756}{279}$	*9.* $\frac{8421}{264}$	*12.* $\frac{14562}{1260}$

39. Reducing Fractions to Least Common Denominator.

—The L. C. D. of two or more fractions is the L. C. M. of their denominators. The process of reduction is that of reducing fractions to higher terms. See Article 35. The values are not changed.

EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$ to L. C. D.

The L. C. D. is 12. Then the object is to reduce the several fractions to 12ths.

Process: (1) $\frac{1}{2}=\frac{6\times1}{6\times2}=\frac{6}{12}$

(2) $\frac{3}{4}=\frac{3\times3}{3\times4}=\frac{9}{12}$

(3) $\frac{5}{6}=\frac{2\times5}{2\times6}=\frac{10}{12}$

2. Reduce $\frac{7}{12}$, $\frac{5}{18}$, $\frac{11}{24}$ to L. C. D.

Process: $\frac{7}{12}$, $\frac{5}{18}$, $\frac{11}{24}$.

(1) The L. C. D. is 72.

(2) $\frac{42}{72}$, $\frac{20}{72}$, $\frac{33}{72}$, result.

(3) $12)\underline{72}$; $7\times6=42$.
$\quad 6$

(4) $18)72$, quotient 4; $5\times4=20$.

(5) $24)72$, quotient 3; $11\times3=33$.

NOTE.—The part on the right, which is the process of finding the numerators, may be put in such form as is convenient. As soon as the L. C. D. is found, the denominators may be placed in (2) thus: $\frac{}{72}$, $\frac{}{72}$, $\frac{}{72}$. Then find the numerators and place them above.

EXERCISE XXIV.

Reduce to L. C. D.:

1. $\frac{2}{7}$, $\frac{9}{14}$, $\frac{11}{28}$.

2. $\frac{5}{34}$, $\frac{9}{17}$, $\frac{13}{85}$.

3. $\frac{22}{25}$, $\frac{17}{75}$, $\frac{11}{50}$.

4. $\frac{6}{11}$, $\frac{37}{55}$, $\frac{13}{22}$, $\frac{19}{44}$.

5. $\frac{13}{30}$, $\frac{17}{15}$, $\frac{8}{45}$, $\frac{43}{60}$.

6. $1\frac{5}{7}$, 5, $\frac{8}{14}$, $\frac{19}{28}$.

NOTE.—Reduce $1\frac{5}{7}$ to an improper fraction before commencing to find L. C. D. Consider 5 as $\frac{5}{1}$.

7. $12\frac{5}{17}$, $95\frac{5}{51}$, $\frac{18}{68}$.

8. $\frac{11}{17}$, $\frac{11}{34}$, $\frac{11}{85}$, $\frac{11}{102}$.

9. $\frac{1}{24}$, $\frac{1}{25}$, $\frac{1}{26}$, $\frac{1}{27}$.

40. Addition of Fractions.—Reduce all fractions to L. C. D., add the numerators, and place the sum over the L. C. D.

EXAMPLES.

1. Add $\frac{2}{3}$, $\frac{4}{5}$, $\frac{1}{6}$.

Process: $\frac{2}{3}+\frac{4}{5}+\frac{1}{6}=\frac{20}{30}+\frac{24}{30}+\frac{5}{30}=\frac{49}{30}=1\frac{19}{30}$, result.

Explanation: Mechanical work necessary to reduce the fractions to L. C. D. need not be preserved. 20 thirtieths+24 thirtieths+5 thirtieths=49 thirtieths, or $\frac{49}{30}$.

NOTE.—If the result is an improper fraction, reduce it to an integer or mixed number.

2. Add $\frac{3}{4}$, $\frac{5}{9}$, $\frac{11}{12}$.

Process: $\frac{3}{4}+\frac{5}{9}+\frac{11}{12}=\frac{27}{36}+\frac{20}{36}+\frac{33}{36}=\frac{80}{36}=2\frac{8}{36}=2\frac{2}{9}$, result.

NOTE.—The $\frac{8}{36}$ should be reduced to its lowest terms, $\frac{2}{9}$. *In results, all fractions should be in their simplest forms.*

3. Add $6\frac{1}{3}$, $10\frac{4}{5}$, $5\frac{4}{15}$.

Process: (1) $\frac{1}{3}+\frac{4}{5}+\frac{4}{15}=$
$\frac{5}{15}+\frac{12}{15}+\frac{4}{15}=$
$\frac{21}{15}=1\frac{6}{15}=1\frac{2}{5}$

(2)
$$\begin{array}{r} 6\frac{1}{3} \\ 10\frac{4}{5} \\ 5\frac{4}{15} \\ \hline 22\frac{2}{5} \end{array}$$

Explanation: The result of adding the fractions is $1\frac{2}{5}$. Write the $\frac{2}{5}$ below the fractions, carry the 1 and add it to the integers; result, $22\frac{2}{5}$.

EXERCISE XXV.

Find the value of—

1. $\frac{11}{17}+\frac{5}{16}$
2. $18\frac{5}{9}+27\frac{7}{12}$
3. $46\frac{8}{15}+\frac{9}{28}$
4. $\frac{35}{51}+\frac{19}{34}+\frac{1}{85}$
5. $\frac{31}{48}+\frac{29}{32}+\frac{17}{24}$
6. $\frac{8}{11}+\frac{5}{9}+\frac{29}{85}+\frac{71}{102}$
7. $18\frac{5}{18}+19\frac{4}{27}+22\frac{7}{81}$
8. $3465+17\frac{9}{24}+\frac{37}{120}$
9. $\frac{121}{144}+1\frac{27}{120}+231$
10. $7\frac{26}{27}+17\frac{80}{81}+9\frac{242}{243}+24\frac{728}{729}$

41. Subtraction of Fractions.—Reduce the fractions to L. C. D. before subtracting. Subtract the numerator of the subtrahend from the numerator of the minuend and place the remainder over the L. C. D.

EXAMPLES.

1. From $\frac{5}{9}$ take $\frac{5}{12}$.

Process: $\frac{5}{9}-\frac{5}{12}=\frac{20}{36}-\frac{15}{36}=\frac{5}{36}$, result.

Explanation: 20 thirty-sixths − 15 thirty-sixths = 5 thirty-sixths, or $\frac{5}{36}$. *In the result, always reduce the fraction to its simplest form.*

2. From $17\frac{1}{3}$ take $14\frac{1}{4}$.

Process: (1) $\frac{1}{3}-\frac{1}{4}=\frac{4}{12}-\frac{3}{12}=\frac{1}{12}$.

$$\begin{array}{r} (2)\ 17\frac{1}{3} \\ \underline{14\frac{1}{4}} \\ 3\frac{1}{12}, \text{ result.} \end{array}$$

3. From $17\frac{1}{7}$ take $5\frac{8}{9}$.

Process: (1) $\frac{8}{7}-\frac{8}{9}=\frac{72}{63}-\frac{56}{63}=\frac{16}{63}$.

$$\begin{array}{r} (2)\ 17\frac{1}{7} \\ \underline{5\frac{8}{9}} \\ 11\frac{16}{63}, \text{ result.} \end{array}$$

Explanation: Since $\frac{8}{9}$ is larger than $\frac{1}{7}$, 1 must be taken from 17 and added to the $\frac{1}{7}$, making $1\frac{1}{7}$ or $\frac{8}{7}$; $\frac{8}{7}-\frac{8}{9}=\frac{16}{63}$. This leaves only 16 in the minuend, from which to subtract the 5.

EXERCISE XXVI.

Find the value of—

1. $\frac{18}{25}-\frac{3}{5}$
2. $7\frac{4}{5}-\frac{11}{27}$
3. $18\frac{5}{12}-\frac{17}{96}$
4. $75-\frac{35}{72}$
5. $37\frac{5}{13}-15\frac{8}{25}$
6. $44\frac{1}{3}-36\frac{9}{10}$
7. $86\frac{15}{19}-\frac{31}{37}$
8. $538\frac{15}{73}-26\frac{11}{95}$
9. $452\frac{1}{20}-247\frac{111}{223}$
10. $140\frac{83}{85}-14\frac{85}{86}$

42. Multiplication of Fractions.—When the multiplier is a fraction, it may be followed by the word "*of*" or the sign, "×." Thus,

$$\frac{3}{5} \text{ of } 10=\frac{3}{5}\times 10.$$

Reduce *mixed numbers* to *improper fractions* before multiplying.

EXAMPLES.

1. Find $\frac{3}{7}$ of 35.

Process Complete: (1) $\frac{1}{7}$ of $35=\frac{35}{7}=5$. (See Art. 16, 3d Ap.)
$3\times(1)=(2)$* $\frac{3}{7}$ of $35=3\times 5=15$, result.

Teacher, show the pupil that (2) comes from (1) by multiplying by 3.

* "$3\times(1)=(2)$" is read *3 times equation one equals equation two.*

Process Shortened: $\dfrac{3\times\overset{5}{\cancel{35}}}{\cancel{7}}=15$, result.

2. Find $\frac{1}{7}$ of $\frac{2}{3}$.

Process:
$\frac{1}{7}$ of $\frac{2}{3}=\frac{2}{3}\div7=\frac{2}{21}$, result.

Explanation: $\frac{2}{3}$ is $2\div3$. Dividing this by 7 is the same as dividing 2 by 3×7, or 21. $2\div21=\frac{2}{21}$.

3. Find $\frac{3}{7}$ of $\frac{2}{3}$.

Process Complete: (1) $\frac{1}{7}$ of $\frac{2}{3}=\frac{2}{21}$.
$3\times(1)=(2)$ $\frac{3}{7}$ of $\frac{2}{3}=3\times\frac{2}{21}=\frac{6}{21}=\frac{2}{7}$, result.

Process Shortened: $\frac{3}{7}$ of $\frac{2}{3}=\dfrac{\cancel{3}\times2}{7\times\cancel{3}}=\frac{2}{7}$, result.

By the *short process*, we express all the numbers as fractions, then multiply all numerators together for a new numerator, and all denominators together for a new denominator. *Employ cancellation.*

4. Find $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{8}{9}$.

Process: $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{8}{9}=\dfrac{1\times\cancel{3}\times\overset{2}{\cancel{8}}}{\cancel{2}\times\cancel{4}\times\underset{3}{\cancel{9}}}=\frac{1}{3}$, result.

5. $7\frac{1}{2}\times\frac{3}{25}=(\quad)$?

Process: $7\frac{1}{2}\times\frac{3}{25}=\dfrac{\overset{3}{\cancel{15}}\times3}{2\times\underset{5}{\cancel{25}}}=\frac{9}{10}$, result.

6. Multiply $\frac{3}{4}$ of 8 by $\frac{1}{3}$ of 6.

Process: $\frac{3}{4}$ of $8\times\frac{1}{3}$ of $6=\dfrac{\cancel{3}\times\overset{2}{\cancel{8}}\times1\times6}{\cancel{4}\times\cancel{3}}=12$, result.

Note.—In the results all improper fractions should be reduced to integers or mixed numbers, and all fractional parts should be reduced to their lowest terms.

EXERCISE XXVII.

Find the value of—

1. $7\times\frac{18}{35}$	*7.* $\frac{6}{13}\times\frac{7}{15}\times\frac{65}{84}$ of $18\frac{3}{4}$
2. $\frac{9}{26}\times 68$	*8.* $5\times 7\frac{3}{10}\times\frac{11}{146}\times 96$
3. $\frac{5}{12}$ of $\frac{72}{95}$	*9.* $126\frac{3}{5}\times\frac{11}{27}\times\frac{81}{101}$ of 633
4. $\frac{5}{9}$ of $\frac{10}{25}$ of 150	*10.* $16\frac{2}{3}\times 30\frac{7}{8}\times 1\frac{15}{16}$
5. $17\frac{5}{14}\times\frac{8}{243}\times 280$	*11.* $127\frac{3}{10}\times 45\frac{5}{9}\times 450$
6. $\frac{3}{13}$ of $2\frac{7}{9}\times\frac{18}{25}$ of $\frac{25}{4}$	*12.* $\frac{32}{51}\times\frac{85}{112}\times\frac{178}{207}\times\frac{23}{72}$

43. Division of Fractions.—*The easiest way to divide in fractions is to invert the divisor and proceed as in multiplication of fractions.* To explain, we know that—

1 apple contains $\frac{1}{2}$ apple 2 times.
1 apple contains $\frac{1}{3}$ apple 3 times.
1 apple contains $\frac{1}{4}$ apple 4 times.
1 apple contains $\frac{2}{4}$ apple 2 times.
1 apple contains $\frac{1}{6}$ apple 6 times.
1 apple contains $\frac{2}{6}$ apple 3 times, etc.

Now, let us arrange these expressions in an abstract form:

$1\div\frac{1}{2}=\frac{2}{1}$, or 2.
$1\div\frac{1}{3}=\frac{3}{1}$, or 3.
$1\div\frac{1}{4}=\frac{4}{1}$, or 4.
$1\div\frac{2}{4}=\frac{4}{2}$, or 2.
$1\div\frac{1}{6}=\frac{6}{1}$, or 6.
$1\div\frac{2}{6}=\frac{6}{2}$, or 3.

Look at each divisor and the first form of the corresponding quotient, and you will see that, in each case, the *quotient* is the *divisor inverted.* This is an illustration of the *general truth : Inverting a divisor shows how many times that divisor is contained in one. Commit this to memory.* With this *truth* in mind, let us study the following examples:

EXAMPLES.

1. Divide 14 by $\frac{7}{8}$.

Process Complete: (1) $1 \div \frac{7}{8} = \frac{8}{7}$. (Above truth.)

$$14 \times (1) = (2) \quad 14 \div \frac{7}{8} = \overset{2}{\cancel{14}} \times \frac{8}{\cancel{7}} = 16, \text{ result.}$$

Process Shortened: $14 \div \frac{7}{8} = \overset{2}{\cancel{14}} \times \frac{8}{\cancel{7}} = 16$, result.

2. $\frac{3}{4} \div \frac{7}{8} = (\quad)$?

Process Complete: (1) $1 \div \frac{7}{8} = \frac{8}{7}$.

$$\tfrac{3}{4} \text{ of } (1) = (2) \quad \frac{3}{4} \div \frac{7}{8} = \frac{3}{\cancel{4}} \text{ of } \frac{\overset{2}{\cancel{8}}}{7} = \frac{6}{7}, \text{ result.}$$

Process Shortened: $\frac{3}{4} \div \frac{7}{8} = \frac{3}{\cancel{4}} \times \frac{\overset{2}{\cancel{8}}}{7} = \frac{6}{7}$, result.

3. $\frac{12}{17} \div 8 = (\quad)$?

Process Complete: (1) $1 \div 8 = \frac{1}{8}$.

$$\tfrac{12}{17} \text{ of } (1) = (2) \quad \frac{12}{17} \div 8 = \frac{\overset{3}{\cancel{12}}}{17} \text{ of } \frac{1}{\underset{2}{\cancel{8}}} = \frac{3}{34}, \text{ result.}$$

Process Shortened: $\frac{12}{17} \div 8 = \frac{\overset{3}{\cancel{12}}}{17} \times \frac{1}{\underset{2}{\cancel{8}}} = \frac{3}{34}$, result.

NOTE.—The first step in the complete process gives us the divisor inverted; this may be done *mentally*. The "*Shortened Process*" is simply the last step in the *Complete Process*, and may be expressed thus: *Invert the divisor and proceed as in multiplication of fractions.* Use the *shorter process* in practice.

4. Divide $7\frac{1}{2}$ by $3\frac{1}{3}$.

Process: $7\frac{1}{2} \div 3\frac{1}{3} = \frac{\overset{3}{\cancel{15}}}{2} \times \frac{3}{\underset{2}{\cancel{10}}} = \frac{9}{4}$, or $2\frac{1}{4}$, result.

5. $\dfrac{12\frac{1}{5}}{1\frac{1}{2}} = (\quad)$?

NOTE.—$\dfrac{12\frac{1}{5}}{1\frac{1}{2}}$ is a complex fraction, and means, like any other fraction, that its numerator, $12\frac{1}{5}$, is to be divided by its denominator, $1\frac{1}{2}$. Then the process is easy:

$$\textit{Process: } \frac{12\frac{1}{5}}{1\frac{1}{2}} = 12\tfrac{1}{5} \div 1\tfrac{1}{2} = \frac{61}{5} \times \frac{2}{3} = \frac{122}{15}, \text{ or } 8\tfrac{2}{15}, \text{ result.}$$

When the terms of the dividend are divisible by the respective terms of the divisor, division of fractions may be performed in that way.

6. $\frac{88}{132} \div \frac{11}{12} = (\quad)$?

$$\textit{Process: } \frac{88}{132} \div \frac{11}{12} = \frac{88 \div 11}{132 \div 12} = \frac{8}{11}, \text{ result.}$$

EXERCISE XXVIII.

Find the value of—

1. $\frac{13}{17} \div 39$	*5.* $27\frac{3}{11} \div 23\frac{1}{13}$	*9.* $\dfrac{15\frac{15}{37}}{18\frac{18}{47}}$
2. $124 \div \frac{31}{44}$	*6.* $18\frac{2}{13} \div 33\frac{5}{7}$	
3. $\frac{540}{941} \div \frac{99}{1882}$	*7.* $\frac{246}{581} \div \frac{84}{2905}$	*10.* $\dfrac{2\frac{4}{11}}{2\frac{2}{5}} \div \dfrac{2\frac{7}{11}}{8\frac{7}{10}}$
4. $18\frac{3}{4} \div 150$	*8.* $\frac{741}{3977} \div \frac{13}{97}$	

44. Greatest Common Divisor of Common Fractions.—The *Greatest Common Divisor* of two or more fractions is the largest fraction that is contained in each of them an integral number of times.

One fraction is a divisor of another, when the numerator of the first is a divisor of the numerator of the second, and the denominator of the first is a multiple of the denominator of the second. Thus,

$\frac{4}{27}$ is a divisor of $\frac{8}{9}$.

This is plain when, according to the process of division of fractions, the divisor is inverted.

$\frac{8}{9} \times \frac{27}{4}$.

The numerator 4 of the divisor is a divisor of the numerator 8 of the dividend, and will disappear; the denominator 27 of the divisor is a multiple of the denominator 9 of the dividend, and the 9 will disappear. When the 4 and 9 both disappear, the quotient will be an integer, and $\frac{4}{27}$ is a *divisor* of $\frac{8}{9}$.

PRINCIPLE: *The Greatest Common Divisor of two or more fractions is that fraction whose numerator is the G. C. D. of the numerators of the several fractions and whose denominator is the L. C. M. of their denominators.*

EXAMPLES.

1. Find the G. C. D. of $\frac{15}{32}$, $\frac{25}{64}$, $\frac{5}{16}$, and $\frac{5}{8}$.

(1) The G. C. D. of 15, 25, 5, 5 = 5.
(2) The L. C. M. of 32, 64, 16, 8 = 64.
$\therefore \frac{5}{64}$ is the required G. C. D.

2. Find the G. C. D. of $14\frac{7}{12}$ and $95\frac{3}{8}$.

(1) Expressed as fractions, $14\frac{7}{12}$ and $95\frac{3}{8} = \frac{175}{12}$ and $\frac{763}{8}$.
(2) The G. C. D. of 175 and 763 = 7.
(3) The L. C. M. of 8 and 12 = 24.
$\therefore \frac{7}{24}$ is the required G. C. D.

EXERCISE XXIX.

Find the G. C. D. of—

1. $\frac{2}{5}$, $\frac{8}{15}$, and $\frac{24}{25}$.
2. $\frac{91}{108}$, $\frac{26}{27}$, $\frac{13}{36}$, and $\frac{39}{59}$.
3. $\frac{12}{19}$, $\frac{28}{57}$, $\frac{4}{9}$, $\frac{64}{513}$, and $\frac{20}{171}$
4. $5\frac{5}{11}$, $12\frac{6}{7}$, and $4\frac{17}{22}$.
5. $9\frac{3}{8}$, $5\frac{5}{6}$, $16\frac{2}{3}$, and $8\frac{1}{3}$.
6. $\frac{6}{7}$, $3\frac{3}{4}$, $\frac{9}{14}$, $1\frac{1}{2}$, and $5\frac{1}{4}$.
7. $3\frac{3}{125}$, $9\frac{4}{5}$, $\frac{14}{15}$, $3\frac{3}{20}$, and $23\frac{1}{3}$.
8. $1\frac{17}{21}$, $\frac{57}{63}$, $10\frac{5}{9}$, $25\frac{1}{3}$, and $16\frac{2}{7}$.
9. $10\frac{5}{56}$, $17\frac{3}{8}$, $12\frac{7}{64}$, $26\frac{3}{4}$, and $18\frac{1}{2}$.
10. $3\frac{5}{6}$, $2\frac{4}{15}$, $12\frac{1}{2}$, $9\frac{3}{4}$, and $8\frac{1}{3}$.

45. Least Common Multiple of Common Fractions.—The Least Common Multiple of two or more fractions is the smallest number that will contain each of them an integral number of times.

One fraction is a multiple of another, when the numerator of the first is a multiple of the numerator of the second, and the denominator of the first is a divisor of the denominator of the second. Thus,

$\frac{8}{9}$ is a multiple of $\frac{4}{27}$.

Indicate the process of division by inverting the divisor. Thus,

$$\frac{8}{9} \times \frac{27}{4}.$$

The 8 is a *multiple* of 4, and the 4 will disappear; the 9 is a *divisor* of 27, therefore the 9 will disappear, and the quotient will be integral.

PRINCIPLE: *The Least Common Multiple of two or more fractions is that fraction whose numerator is the L. C. M. of the numerators of the several fractions, and whose denominator is the G. C. D. of their denominators.*

EXAMPLES.

1. Find the L. C. M. of $\frac{3}{4}$, $\frac{9}{16}$, and $\frac{21}{32}$.

(1) The L. C. M. of 3, 9, 21 = 63.
(2) The G. C. D. of 4, 16, 32 = 4.
∴ $\frac{63}{4}$ or $15\frac{3}{4}$ is the required L. C. M.

2. Find the L. C. M. of $14\frac{2}{7}$, $9\frac{1}{11}$, $16\frac{2}{3}$, and 25.

(1) The numbers expressed as fractions are $\frac{100}{7}$, $\frac{100}{11}$, $\frac{50}{3}$, $\frac{25}{1}$.
(2) The L. C. M. of 100, 100, 50, 25 = 100.
(3) The G. C. D. of 7, 11, 3, 1 = 1.
∴ $\frac{100}{1}$ or 100 is the required L. C. M.

EXERCISE XXX.

Find the L. C. M. of—

1. $\frac{7}{90}$, $\frac{22}{105}$, and $\frac{11}{60}$.
2. $\frac{5}{8}$, $\frac{9}{32}$, $\frac{15}{16}$, and $\frac{1}{4}$.
3. $\frac{3}{5}$, $\frac{9}{50}$, $\frac{6}{25}$, and $\frac{12}{35}$.
4. $33\frac{1}{3}$, $12\frac{1}{2}$, $6\frac{2}{3}$, and $7\frac{2}{9}$.
5. $1\frac{11}{49}$, $1\frac{13}{35}$, $\frac{120}{343}$, $\frac{4}{63}$, and $\frac{2}{77}$.
6. $\frac{9}{10}$, $3\frac{3}{50}$, $4\frac{8}{25}$, $12\frac{3}{5}$, and $4\frac{1}{20}$.
7. $18\frac{2}{3}$, $15\frac{3}{4}$, $19\frac{2}{8}$, and $4\frac{1}{5}$.
8. $37\frac{1}{2}$, $1\frac{7}{8}$, $8\frac{1}{3}$, $10\frac{5}{7}$, and $18\frac{3}{4}$.
9. $22\frac{8}{11}$, $33\frac{1}{3}$, $62\frac{1}{2}$, $5\frac{5}{9}$, and $166\frac{2}{3}$.
10. $142\frac{6}{7}$, $55\frac{5}{9}$, $41\frac{2}{3}$, and $333\frac{1}{3}$.

J. DECIMAL FRACTIONS.

46. Definitions.—A **Decimal Fraction** has for its expressed denominator 1 with 0's annexed. Thus:

$$\tfrac{1}{10},\ \tfrac{5}{100},\ \tfrac{124}{1000},\ \tfrac{2045}{10000}.$$

A decimal fraction may be written without using figures for its denominator. Thus:

$$\tfrac{1}{10}=.1.$$
$$\tfrac{5}{100}=.05.$$
$$\tfrac{124}{1000}=.124.$$
$$\tfrac{2045}{10000}=.2045.$$

By this method of writing decimal fractions, (1) the numerator is written just as an integer; (2) a period, called a **Decimal Point,** is put on the left of as many places in the numerator as there are 0's in the expressed denominator. If there are not so many places in the numerator as are needed, 0's are prefixed to the numerator to make the required number of places.

A decimal fraction written in this way is called a **Decimal.**

The denominator of a decimal—

of one place is 10; .7 is read $\tfrac{7}{10}$;
of two places is 100; .46 is read $\tfrac{46}{100}$;
of three places is 1000; .307 is read $\tfrac{307}{1000}$.
of four places is 10000; .0453 is read $\tfrac{453}{10000}$; and so on.

The decimal places are named from the decimal point to the right, as follows:

The first place, **tenths**;
the second place, **hundredths**;
the third place, **thousandths**;
the fourth place, **ten-thousandths**;
the fifth place, **hundred-thousandths**; and so on.

The decimal orders are numbered from tenths to the right as the integral orders are numbered from units to the left. Thus,

	Integral Places.							Decimal Places.				
Places	hundred-thousands.	ten-thousands.	thousands.	hundreds.	tens.	units.		tenths.	hundredths.	thousandths.	ten-thousandths.	hundred-thousandths.
Digits	6	4	5	3	7	2	.	3	8	9	1	7
Orders	6th	5th	4th	3d	2d	1st		1st	2d	3d	4th	5th
	Integral Orders.							Decimal Orders.				

The law of position (or order) in decimals is the same as that in integers.

LAW: *Ten units of any one order make one unit of the next higher order.*

The denominator of a decimal is indicated by the name of its right-hand place. Thus, in

.0246,

the name of the right-hand place is *ten-thousandths;* the denominator is *ten thousand.*

When a decimal is annexed to an integer the expression is called a **Mixed Decimal.** Thus:

5.04; and is read, $5\frac{4}{100}$.

A common fraction may be annexed to a decimal; such an expression is called a **Complex Decimal.** Thus:

$$.12\tfrac{1}{2}\text{; and is read, } \frac{12\frac{1}{2}}{100}.$$
$$5.37\tfrac{1}{2}\text{; and is read, } 5\frac{37\frac{1}{2}}{100}.$$

In an expression containing two or more decimal points all decimal points may be omitted except the one farthest to the left. Thus:

$$59.6.4 = 59.64.$$

Explanation: $59.6.4 = 59.6\frac{4}{10} = 59\frac{6\frac{4}{10}}{10} = 59\frac{\frac{64}{10}}{10} = 59\frac{64}{100} = 59.64.$

There is no such expression in our system of notation as a common fraction with a decimal point on the left of it. Thus,

$$.\tfrac{1}{3}.$$

Explanation: $\frac{1}{3}$ without the point is $\frac{1}{3}$ of one integral unit. $.0\frac{1}{3}$ is $\frac{1}{3}$ of of one-tenth, which is the first decimal order. Therefore there is no place in our system for such an expression as $.\frac{1}{3}$.

EXERCISE XXXI.

Read:

1. .6	*6.* .049	*11.* 100.00001
2. .25	*7.* 51.346	*12.* .04567
3. .08	*8.* .00.01	*13.* $17.37\frac{1}{2}$
4. .73	*9.* 7.0834	*14.* $.065\frac{2}{4}\frac{5}{9}$
5. 12.5	*10.* .759.63	*15.* $143.00\frac{3}{5}\frac{7}{9}$

Write as decimals:

16. $.\frac{7}{10}$	*21.* $\frac{247}{1000}$	*26.* $59\frac{\frac{3}{11}}{10000}$
17. $\frac{8\frac{1}{3}}{10}$	*22.* $\frac{\frac{5}{7}}{1000}$	*27.* $100\frac{1}{1000000}$
18. $\frac{\frac{1}{3}}{10}$	*23.* $\frac{8\frac{1}{3}}{1000}$	*28.* $11\frac{11111\frac{1}{11}}{100000}$
19. $\frac{87}{100}$	*24.* $12\frac{12}{100}$	*29.* $375\frac{486\frac{5}{12}}{100000}$
20. $.\frac{87\frac{1}{3}}{100}$	*25.* $47\frac{\frac{1}{3}}{10}$	*30.* $304\frac{304\frac{5}{9}}{10000}$

47. Reducing Decimals to Higher Terms.—If a 0 is annexed to a decimal, the numerator is multiplied by 10; but since another decimal place is thus added, the denominator is also multiplied by 10, and the value of the fraction is not changed. Thus:

$$.5 = .50 = .500 = .5000, \text{ etc.}$$

$$\text{That is, } \tfrac{5}{10} = \tfrac{50}{100} = \tfrac{500}{1000} = \tfrac{5000}{10000}, \text{ etc.}$$

EXAMPLE.

Reduce .125 to 100000ths.

Process: .125 = .12500, result.

Explanation: Annex 0's to the decimal till the number of decimal places is equal to the number of 0's in the required denominator.

EXERCISE XXXII.

Reduce:

1. .7 to 1000ths.
2. .506 to 10000ths.
3. .94 to 1000ths.
4. .08 to 10000ths.
5. .004 to 10000ths.
6. 5.1 to 100ths.
7. 7.05 to 1000ths.
8. 56.084 to 10000ths
9. 15* to 1000ths.
10. 14.506 to 100000ths.

48. Reducing Decimals to Lower Terms.—If there are 0's at the right of a decimal, they may be dropped without changing the value of the decimal. Thus:

$$.2300 = .23; \text{ that is,}$$

$$\tfrac{2300}{10000} = \tfrac{23}{100}.$$

EXAMPLES.

1. Reduce .750 to 100ths.

Process: .750 = .75, result.

2. Reduce .24000 to 1000ths.

Process: .24000 = .240, result.

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* Simply place the decimal point after the 15 and add the 0's.

EXERCISE XXXIII.

Reduce —

1. .200 to 10ths.
2. .3400 to 100ths.
3. .10500 to 1000ths.
4. .50000 to 100ths.
5. .94000 to 100ths.
6. 5.0000 to an integer.
7. 7.5000 to 10ths.
8. 15.0700 to 100ths.
9. 70.0700 to 100ths.
10. .005000 to 1000ths.

49. Addition of Decimals.—Write the numbers to be added so that the decimal points will form a column. *Add as in integers, placing the decimal point in the result beneath the column of decimal points above.*

EXAMPLE.

1. Add .501, 3.45, .125, 7.0034, 18.3024.

Long Way.	*Short Way.*
.5010	.501
3.4500	3.45
.1250	.125
7.0034	7.0034
18.3024	18.3024
29.3818, result.	29.3818, result.

Explanation: In the *long way*, all decimals are reduced to the highest denomination, 10000ths. The denomination of the sum is 10000ths. Why?

In the *short way*, the decimals are not reduced to higher denominations. The 0's on the right do not affect the result. Use the *short way* in practice.

EXERCISE XXXIV.

Add:

1. .52, .936, 1.63, .0075.
2. .428, .506, .362, .521, .736.
3. .002, 5.3043, 7.5555, 81.8008
4. 12.342, 18.9813, 85.753, .00059.
5. 8.756, .00045, .04621, .989898, .010101.
6. $25.30, $846.27, $954.63, $1205.84, $746.98, $1375.30, $3.85, $840.75, $1684.35.

NOTE.—When an expression of $'s has only two decimal places, call the decimal part *cents;* if there are more than two decimal places, read the decimal part as a fraction of a dollar.

7. \$44.56, \$752.28, \$984.31, \$1200.50, \$845.95, \$7400.20, \$947.25, \$75.75.
8. \$56.345, \$824.955, \$5476.375, \$1260, \$95.625, \$4000.50, \$1862.05.
9. 586.034, 7495.6034, 845.9567, 1111.1111, 9988.7766, 1200.0001, .99864.
10. .04654, .90083, .12456, .00665, .77777, .854321, .750864, .999999.

50. Subtraction of Decimals.—Write the numbers so that the decimal point in the subtrahend will be beneath the decimal point in the minuend. *Subtract as in integers, placing the decimal point in the result beneath the decimal points above.*

EXAMPLES.

1. From 5.47 take 2.638.

Process:

5.470
2.638
2.832, result.

Explanation: Reduce the minuend to the denomination of the subtrahend by annexing 0 to it. The denomination of the result is 1000ths. Why?

2. From .54652 take .369.

Long Way.

.54652
.36900
.17752, result.

Short Way.

.54652
.369
.17752, result.

Explanation: In the *long way*, the subtrahend is reduced to the denomination of the minuend. The denomination of the result is 100000ths. Why? In the *short way*, the subtrahend is not reduced to higher denominations. The 0's on the right do not affect the result. Use the *short way* in practice.

EXERCISE XXXV.

Subtract:

1. .2464 from .65.
2. .38 from 5.001.
3. .428 from .6.
4. .5743 from .8046.
5. 1.5644 from 5.
6. 3.427 from 8.245.
7. .0347 from .5037.
8. .0042 from .042.
9. \$1.20 from \$7.05.
10. \$246.75 from \$4000.

51. Multiplication of Decimals.—*Multiply as in integers, neglecting 0's that may be on the left of the multiplicand and multiplier, and place the decimal point in the result so that there will be as many decimal places in the product as there are in both multiplicand and multiplier.*

EXAMPLES.

1. Multiply .852 by .35.

Process.

$$\begin{array}{r}
.852 \\
\underline{.35} \\
4260 \\
\underline{2556} \\
.2982\cancel{0}
\end{array}\text{, result.}$$

Explanation: The multiplicand is *thousandths;* the multiplier is *hundredths.* Hundredths times thousandths gives hundred-thousandths. Show this by use of common fractions. Hundred-thousandths is expressed by a decimal of 5 places. After placing the decimal point, cancel the 0's on the right; for the product should be expressed in its lower terms.

2. Multiply .0037 by 2.5.

Process.

$$\begin{array}{r}
.0037 \\
\underline{2.5} \\
185 \\
\underline{74} \\
.00925
\end{array}\text{, result.}$$

Explanation: For the purpose of the multiplication, the multiplicand is 37 (neglecting the 0's on the left).

The product after multiplying contains but *three* figures; but there must be *five* decimal places. Why? Therefore two 0's are prefixed and the decimal point is placed to the left.

3. Multiply 246 by 3.07.

Process.

$$\begin{array}{r}
246 \\
\underline{3.07} \\
1722 \\
\underline{738} \\
755.22
\end{array}$$

NOTE.—Have pupil explain.

EXERCISE XXXVI.

Multiply —

1. .75 by .23.
2. .428 by .501.
3. 4.763 by .034.
4. .00073 by .0059.
5. .5064 by .00001.
6. 56.043 by .059.
7. 75.003 by 5.006.
8. 1000.001 by 100.0001.
9. 1111.22 by 5.060701.
10. $420.50 by .37.
11. $4860.05 by 246.
12. $840.56 by 10.05.

52. Division of Decimals.—*Divide as in integers. neglecting 0's that may be on the left of the dividend or divisor.*

Since the dividend is equal to the product of the divisor and quotient (Art. 14, Prin. 3), the dividend will have as many decimal places as both divisor and quotient. From this, the following conclusions are evident:

(1) *The dividend must not have fewer decimal places than the divisor.* (Commit.)

NOTE.—If the dividend has not as many decimal places as the divisor, enough 0's to make them equal should be annexed to the dividend before dividing.

EXAMPLE.

Prepare to divide .15 by .0025.

Form.

$.0025\overline{).1500}$

Explanation: Two 0's must be annexed to the dividend, because there are four decimal places in the divisor and only two decimal places in the dividend.

(2) *If, after dividing, the number of decimal places used in the dividend equals the number in the divisor, the quotient is an integer.* (Commit.)

EXAMPLE.

Divide .15 by .0025.

Process.

```
         60, result.
.0025).1500
        150
        ---
          0
```

NOTE.—In the actual process of division, the two 0's on the left of the divisor are not used; the divisor is 25.

(3) *If, after dividing, the number of decimal places used in the dividend exceeds the number of decimal places in the divisor, the number of decimal places in the quotient must equal that excess.* (Commit.)

EXAMPLES.

1. Divide .98745 by .29.

```
       3.4 0 5, result.
.2 9).9 8 7 4 5
      8 7
      ---
      1 1 7
      1 1 6
      -------
          1 4 5
          1 4 5
          -----
```

Explanation: There are *five* decimal places in the dividend and *two* in the divisor. The *excess* is three. Therefore, the quotient must contain *three decimal places.*

2. Divide .002868 by .239.

Process.

```
        .012, result.
.239).002868
        239
        ---
         478
         478
         ---
```

Explanation: After dividing the 2868 (neglecting the 0's on the left) by 239, the quotient is 12; but the *excess* of decimal places in the dividend over those in the divisor is *three.* Therefore, the quotient must contain *three decimal places.*

If there is a remainder after using the last figure of the dividend, 0's may be annexed to the dividend and the division continued at pleasure.

EXAMPLE.

Divide 2.09 by .017.

Process.

```
         122.94117+, result.
.017)2.09000000
     17
     --
      39
      34
      --
       50
       34
       ---
       160
       153
       ---
         70
         68
         --
          20
          17
          --
           30
           17
           ---
           130
           119
           ---
            11
```

Explanation: The plus sign after the quotient shows that there is a remainder, and that the division could be continued farther, if desired. After dividing as far as desired, locate the decimal point according to Rule 3.

EXERCISE XXXVII.

Divide—

1. .125 by .5.	*7.* .025 by 5.
2. .5 by 1.28.	*8.* .025 by .5.
3. .12 by .0016.	*9.* 1000 by .001.
4. .12 by 16.	*10.* .5484888 by .0067.
5. 12 by .16.	*11.* $4326.422 by $96.1.
6. .001 by 1000.	*12.* 104.8576 by .001024.

53. Reducing Common Fractions to Decimals.—Since a common fraction is an expression of division, if the division be performed, the result will be an integer or a decimal. In dividing, observe the rules for division of decimals.

EXAMPLES.

1. Reduce $.7\frac{3}{16}$ to a decimal.

Process.

$$\begin{array}{r|l} & .1875 \\ 16 & 3.0000 \\ & \underline{16} \\ & 140 \\ & \underline{128} \\ & 120 \\ & \underline{112} \\ & 80 \\ & \underline{80} \end{array}$$

.7.1875 = .71875, result.

Explanation: By reducing the $\frac{3}{16}$ to a decimal and annexing it to the .7, the result is .7.1875. But all decimal points may be omitted except the one on the left.

2. Reduce $\frac{7}{33}$ to a decimal. (Continued 4 places.)

Process.

$$\begin{array}{r|l} & .2121+\text{, result.} \\ 33 & 7.0000 \\ & \underline{66} \\ & 40 \\ & \underline{33} \\ & 70 \\ & \underline{66} \\ & 40 \\ & \underline{33} \\ & 7 \end{array}$$

Explanation: It will be observed that this division will never be complete; for 7 was the figure in the dividend at the beginning. It was the remainder after the 2d division, and also after the 4th division. In such a process, the division will never be complete, if the denominator has any other factor than 2 or 5. These are factors of 10, and by adding 0's to the numerator we multiply by 10. The numerator can thus be made to contain any number of 2's or 5's, but no other factor.

In such a decimal as is obtained in example 2 above, one or more figures continue to recur in a certain order. Such decimals are called **Recurring** or **Circulating Decimals**, or simply **Circulates.** The part that recurs is called the **Repetend,** and is marked by placing a dot over the first and last figures of the repetend. Thus,

$$\dot{5}, \qquad .2\dot{1}3\dot{4}.$$

The first, if written out, would be —

.5555555..........to infinity.

Note.—"*To infinity*" means that the expression would never terminate, but would continue forever as indicated.

The second, if written out, would be —

.2134134134134......to infinity.

3. Reduce $\frac{16}{37}$ to a circulating decimal.

Process.

$$
\begin{array}{r@{}l}
 & .\dot{4}\,3\,\dot{2},\ \text{result.} \\
37) & 16.000 \\
 & 148 \\
\hline
 & \ \ 120 \\
 & \ \ 111 \\
\hline
 & \ \ \ \ 90 \\
 & \ \ \ \ 74 \\
\hline
 & \ \ \ \ 16
\end{array}
$$

Explanation: After three divisions the remainder is *16*, which is the same as at the start. Therefore the repetend consists of 3 figures.

4. Reduce $\frac{7}{22}$ to a circulate.

Process.

$$
\begin{array}{r@{}l}
 & .3\,\dot{1}\,\dot{8},\ \text{result.} \\
22) & 7.000 \\
 & 66 \\
\hline
 & \ \ 40 \\
 & \ \ 22 \\
\hline
 & \ \ 180 \\
 & \ \ 176 \\
\hline
 & \ \ \ \ \ 4
\end{array}
$$

Explanation: The same remainder recurs after the first and third divisions; therefore the second and third divisions will be repeated as long as the division is continued, and the repetend is 18, commencing with the second decimal place.

EXERCISE XXXVIII.

Reduce to decimals, obtaining complete results :

1. $\frac{7}{8}$	*4.* $12\frac{34}{80}$	*7.* $73\frac{41}{128}$
2. $\frac{11}{25}$	*5.* $43\frac{11}{64}$	*8.* $.5\frac{5}{8}$
3. $\frac{17}{32}$	*6.* $3.7\frac{1}{40}$	*9.* $.06\frac{11}{20}$

Reduce to decimals, obtaining results true to four places :

10. $\frac{5}{14}$	*13.* $\frac{19}{34}$	*16.* $.07\frac{59}{218}$
11. $\frac{13}{42}$	*14.* $12\frac{16}{63}$	*17.* $34.9\frac{10}{113}$
12. $\frac{7}{24}$	*15.* $8.4\frac{3}{7}$	*18.* $741.07\frac{71}{82}$

Reduce to circulating decimals:

19. $\frac{5}{9}$	*22.* $\frac{5}{11}$	*25.* $12\frac{472}{555}$
20. $\frac{11}{24}$	*23.* $\frac{104}{165}$	*26.* $\frac{4}{165}$
21. $\frac{1}{7}$	*24.* $\frac{23}{45}$	*27.* $15\frac{13}{74}$

54. Reducing Decimals to Common Fractions.—Express the denominator, which is *1* with as many 0's annexed as there are decimal places. Then reduce the result to its simplest form.

EXAMPLES.

1. Reduce .15 to a common fraction.

Process: $.15 = \frac{15}{100} = \frac{3}{20}$, result.

2. Reduce .0036 to a common fraction.

Process: $.0036 = \frac{36}{10000} = \frac{9}{2500}$, result.

3. Reduce 5.125 to a mixed number.

Process: $.125 = \frac{125}{1000} = \frac{1}{8}$.

$\therefore$ $5\frac{1}{8}$ is the required result.

NOTE.—Reduce the decimal part to a common fraction and annex it to the integer.

4. Reduce $.12\frac{1}{2}$ to a common fraction.

Process: $.12\frac{1}{2} = \frac{12\frac{1}{2}}{100} = 12\frac{1}{2} \div 100 = \frac{25}{2} \times \frac{1}{100} = \frac{1}{8}$.

5. Reduce $.\dot{6}$ to a common fraction.

Process: (1) $.\dot{6} = .666666$ .... to infinity.

$10 \times (1) = (2)$ $10 \times .\dot{6} = 6.666666$ ...... to infinity.

$(2) - (1) = (3)$ $9 \times .\dot{6} = 6$.

$\frac{1}{9}$ of $(3) = (4)$ $.\dot{6} = \frac{6}{9} = \frac{2}{3}$, result.

NOTE.—Subtracting the repetend from 10 times the repetend leaves 9 times the repetend. Subtracting .666666 to infinity from 6.666666 to infinity leaves 6. Therefore 9 times the repetend equals 6, and the repetend equals $\frac{2}{3}$.

6. Reduce $.\dot{5}0\dot{7}$ to a common fraction.

Process: (1) $.\dot{5}0\dot{7} = .507507507............$ **to infinity.**

$1000 \times (1) = (2)\ 1000 \times .\dot{5}0\dot{7} = 507.507507......$ **to infinity.**

$(2) - (1) = (3)\ 999 \times .\dot{5}0\dot{7} = 507.$

$\frac{1}{999}$ **of** $(3) = (4)\ \dot{5}0\dot{7} = \frac{507}{999} = \frac{169}{333}$, **result.**

PRINCIPLE: *Any repetend may be expressed as a common fraction by writing the repetend for the numerator and as many 9's for the denominator as there are figures in the repetend.*

7. Reduce $17.\dot{5}\dot{4}$ to a mixed number.

Process: $.\dot{5}\dot{4} = \frac{54}{99} = \frac{6}{11}.$

$\therefore\ 17.\dot{5}\dot{4} = 17\frac{6}{11}$, **result.**

8. Reduce $.25\dot{1}0\dot{8}$ to a common fraction.

Process: (1) $.25\dot{1}0\dot{8} = .25\frac{108}{999} = .25\frac{12}{111}.$

(2) $.25\frac{12}{111} = \dfrac{25\frac{12}{111}}{100} = \dfrac{\frac{2787}{111}}{100} = \frac{2787}{11100} = \frac{929}{3700}$, **result.**

NOTE.—In processes where circulating decimals are involved, reduce the circulate to a common fraction before beginning the process. At the conclusion, the result may be expressed as a circulate, if required.

EXERCISE XXXIX.

Reduce to common fractions or mixed numbers:

1. .375	*7.* $.4\frac{1}{2}$	*13.* $.\dot{6}\dot{3}$
2. .125	*8.* $.16\frac{2}{3}$	*14.* $.0\dot{6}\dot{3}$
3. .425	*9.* $.47\frac{7}{8}$	*15.* $.24\dot{8}$
4. 5.625	*10.* $.037\frac{1}{21}$	*16.* $.5\dot{7}\dot{2}$
5. 8.875	*11.* $2.33\frac{1}{33}$	*17.* $.1\dot{2}5\dot{2}$
6. .025	*12.* $15.84\frac{51}{121}$	*18.* $7.0\dot{1}57\dot{5}$

II. STUDY OF PROBLEMS.

A. THE MEANS OF EXPRESSING SOLUTIONS.

1. EQUATIONS.

55. Definitions.—A statement, in which the sign of equality (=) is used between two numerical expressions to show that they are equal, is an **Equation.** The expression on the left of the sign is called the *first* or *left member;* that on the right, the *second* or *right member.* When a member of an equation is separated into parts by either of the signs, plus (+) or minus (−), or both, these parts are called **Terms;** when not so separated, the whole member is called a term. See *Terms*, Article 21.

EXAMPLES.

1. $9-4+5=10$.

9, 4, 5, 10, are terms.

2. A's money − \$50 = B's money.

A's money, \$50, and B's money, are terms.

3. 4 pecks = 1 bushel.

Each member is a term.

4. Area of a surface 1 foot long, 1 foot wide = 1 square foot.

Each member is a term.

56. Classification as to Source.—Some equations express constant relations, and are always true; other equations express relations depending upon conditions, and are therefore true for particular conditions only.

The following are some of the equations which are always true:

1. $6+4-7=3$.
2. $5\times8=40$.
3. $56\div8=7$.
4. 4 pecks = 1 bushel.
5. Area of a surface 1 foot long, 1 foot wide = 1 square foot.
6. 8 × B's money + 3 × B's money = 11 × B's money.

The following equations are true for particular conditions only:

1. Cost of 5 books=\$8.50.
2. $\frac{1}{2}$ of A's age=B's age+6 years.
3. Weight of 1 load=2475 pounds.
4. Money earned by 7 men in 2 days=\$35.

NOTE.—As equations of the first class are always true, they may be used in the solution of any problem where such use is of service; but those of the second class only when by the conditions of the problem they are true.

57. Transformation.—An equation may be changed, or transformed from a given to a required form, by subjecting it to one or more of the following processes:

1. To turn an equation around.
2. To transpose the terms of an equation.
3. To simplify the members of an equation.
4. To multiply an equation.
5. To divide an equation.

58. To Turn an Equation Around.—To turn an equation around is to put the left member on the right and the right member on the left of the sign of equality.

PRINCIPLE: *Equals are equal in whatever order considered.*

It follows from this principle that any equation may be turned around and still remain a true equation.

EXAMPLES.

1. 4 gills=1 pint; then, 1 pint=4 gills.
2. Cost of 5 hats=\$15; then, \$15=cost of 5 hats.
3. \$740−cost price=\$560; then, \$560=\$740−cost price.

NOTE.—This process is so easy that no exercise is given for practice.

59. Transposition of Terms.—Transposing a term is changing it from one member of an equation to the other.

NOTE.—Transposition is looked upon by many pupils as beyond the sphere of arithmetic. The following exercise is given with the aim of suggesting that transposition depends upon the most elementary knowledge of number and not upon some difficult mathematical principle.

EXERCISE XL.

Answer each question, then tell what number has been transposed :

1. $5+6=11$; then, $5=11-(\ \ )$?
2. $5+6=11$; then, $6=11-(\ \ )$?
3. $9-5=4$; then, $9=4+(\ \ )$?
4. $9=6+3$; then, $9-(\ \ )=6$?
5. $7-4+9=12$; then, $7+9=12+(\ \ )$?
6. $7-4+9=12$; then, $7-4=12-(\ \ )$?
7. $7-4+9=12$; then, $9-4=12-(\ \ )$?
8. $5+4=12-3$; then, $5+4+(\ \ )=12$?
9. $14-8=4+2$; then, $14-8-(\ \ )=2$?

PRINCIPLES: *1. If equals be added to equals, the sums will be equal.*

2. If equals be subtracted from equals, the remainders will be equal.

EXAMPLES.

1. \$75−\$20=\$25+\$30; transpose \$20.

Process : (1) $\$75-\$\cancel{20}=\$25+\$30.$
(2) $\$\cancel{20}=\$20.$
(1)+(2)=(3) $\$75=\$25+\$30+\$20.$

Explanation : If (2) be added to (1), the sum of the first members will be \$75−\$20+\$20. The two 20's, one to be subtracted and the other to be added, balance each other and may be canceled. This leaves \$75 for the first member of (3). The second members of (1) and (2) added give \$25+\$30+\$20.

2. A's money + \$50 = B's money; transpose A's money.

Process: (1) ~~A's money~~ + \$50 = B's money.
(2) ~~A's money~~ = A's money.
(1) − (2) = (3) \$50 = B's money − A's money.

NOTE.—Let the pupil explain.

3. \$40 − \$30 = \$60 − \$50; transpose the \$50.

Process: (1) \$40 − \$30 = \$60 − ~~\$50~~.
(2) \$50 = ~~\$50~~.
(1) + (2) = (3) \$40 − \$30 + \$50 = \$60.

NOTE.—Let the pupil explain.

Each of these exercises and examples has illustrated one principle, which may be called the *law of transposition.*

LAW.—*Any term of one member of an equation may be transposed to the other member, if at the same time its sign is changed, the plus to minus or the minus to plus.*

4. $30 - 15 + 40 = 66 - 11$; transpose 11 and 15.

Result: $30 + 40 + 11 = 66 + 15$.

NOTE.—The *process* should be mental.

5. $51 - 12 + 5 = 25 + 19$. Transpose the 12 and the 19.

Result: $51 + 5 - 19 = 25 + 12$.

6. 100% of cost + \$415 = $36\frac{1}{2}$% of cost + \$1050. Transpose so that only similar terms will be in the same member.

Result: 100% of cost − $36\frac{1}{2}$% of cost = \$1050 − \$415.

NOTE.—The sign, %, is read *per cent;* and it means *hundredth* or *hundredths,* according as it is used with one or more than one.

EXERCISE XLI.

1. Cost + \$200 = 3 × cost + \$80. Transpose *\$'s* to the first member and *cost* to the second.

2. Cost of 15 hats = \$70 + cost of 1 hat. Leave *\$'s* only in second member.

3. $25-17+44-19=36-15+12$. Transpose so as to leave no minus signs.

4. 75% of No. $-200=$ 40% of No. $+200$. Transpose so that *similar* terms will be together, *per cent* in the first member.

5. $\frac{1}{4}$ of the tens $+\frac{1}{3}$ of the units $=\frac{1}{2}$ of the units. Put all the *units* in the second member.

6. A's money $-\$600=\frac{1}{2}$ of A's money $-\$200$. Transpose so as to leave no minus signs.

7. 10 weeks' wages $-\$12=4$ weeks' wages $+\$24$. Transpose so that similar terms will be together, *$'s* in the second member.

8. $5\times$ my age $+24$ years $=3\times$ my age $+104$ years. Transpose so that similar terms will be together, *years* in the second member.

60. To Simplify Members.—A member of an equation is a *numerical expression*. By transposition, the terms of any member may be made similar. A member whose terms are similar may be simplified by the same processes that are employed to simplify other numerical expressions. See Article 21.

EXAMPLE.

1. $\frac{1}{2}$ of wt. $+\frac{1}{3}$ of wt. $+\frac{3}{4}$ of wt. $=6$ pounds $+5$ pounds $+8$ pounds.

Process: (1) $\frac{1}{2}$ of wt. $+\frac{1}{3}$ of wt. $+\frac{3}{4}$ of wt. $=6$ pounds $+5$ pounds $+8$ pounds.
(1) simplified $=$ (2) $\frac{19}{12}$ or $1\frac{7}{12}\times$ wt. $=19$ pounds.

EXERCISE XLII.

Simplify the members of the resulting equations in Exercise XLI.

NOTE.—Before attempting to simplify, be sure that similar terms are in the same member.

61. To Multiply an Equation.—An equation is multiplied by multiplying each of its members by the same number.

GENERAL PRINCIPLES: *1. Equals multiplied by the same number give equal products.*

2. Multiplying each term of a member of an equation by the same number multiplies the member by that number.

NOTE.—In addition to these general principles, a number of special principles are illustrated in the following examples. They will be given in connection with their respective illustrations, and should be committed to memory.

EXAMPLES.

1. 1 bushel=4 pecks. Multiply by 5.

Process: (1) 1 bu. = 4 pk.
*5×(1)=(2) 5 bu. = 20 pk.

2. My age=35 years. Multiply by 12.

Process: (1) My age = 35 yr.
12×(1)=(2) 12×my age = 420 yr.

3. Cost of 1 hat=\$2. Multiply by 8. (2 forms.)

Process: (1) Cost of 1 hat = \$2.
8×(1)=(2) 8×cost of 1 hat = \$16; or,
8×(1)=(3) Cost of 8 hats = \$16.

NOTE.—Observe that either equation (2) or (3) is the required form, and that *8×cost of 1 hat = cost of 8 hats.*

PRINCIPLES: *1. The cost of any number of articles of the same kind is equal to that number of times the cost of 1 of those articles.*

4. 4+9−3+6=16. Multiply by 7.

Process: (1) 4+9−3+6=16.
7×(1)=(2) 28+63−21+42=112. (Prin. 2.)

5. \$50−A's money=½ of B's money. Multiply by 8.

Process: (1) \$50−A's money = ½ of B's money.
8×(1)=(2) \$400−8×A's money = 4×B's money.

*"5×(1)=(2)" is read *5 times equation 1 equals equation 2.*

6. $6 \times 5 = 30$. Multiply by 4. (2 forms.)

Process: (1) $6 \times 5 = 30$.
$4 \times (1) = (2)$ $(4 \times 6) \times 5 = 24 \times 5 = 120$; or,
$4 \times (1) = (3)$ $6 \times (4 \times 5) = 6 \times 20 = 120$. (Prin. 1, Art. 27.)

7. Area of a surface 1 foot long, 1 foot wide = 1 square foot. Multiply by 6. (2 forms.)

Process: (1) Area of a surface 1 ft. l., 1 ft. w. = 1 sq. ft.
$6 \times (1) = (2)$ Area of a surface 6 ft. l., 1 ft. w. = 6 sq. ft.; or,
$6 \times (1) = (3)$ Area of a surface 1 ft. l., 6 ft. w. = 6 sq. ft

NOTE 1.—In general, *length* means the longest dimension of a surface or solid, but mathematically it simply means *one* of the two or three dimensions, and it is not necessarily the longest.

NOTE 2.—The equation in the above example expresses a certain *area* —"*Area of a surface (of certain dimensions) = 1 sq. ft.*" Any process which multiplies the area of this surface, multiplies this equation. This can be done by multiplying either the *length* or the *width*, but not both. A surface 6 times as long and of the same width will have 6 times as much area, or a surface of the same length but 6 times as wide will have 6 times as much area.

2. (1) *Multiplying the length or width multiplies the area; therefore*, (2) *the length and width bear to the area the same numerical relation as that of factors to the product.*

NOTE.—"*Numerical relation*" means relation as a number without considering its denomination or kind. Two numbers expressing feet (as, 4 feet and 5 feet) could not be factors, could not be multiplied together, but their *numerical values* (4 and 5) could be multiplied together and could be factors.

Question: Does Prin. 1, Art. 27, apply to example 7?

8. The volume of a solid 1 foot long, 1 foot wide, and 1 foot thick = 1 cubic foot. Multiply by 12. (3 forms.)

Process: (1) Vol. of a solid 1 ft. l., 1 ft. w., 1 ft. th. = 1 cu. ft.
$12 \times (1) = (2)$ Vol. of a solid 12 ft. l., 1 ft. w., 1 ft. th. = 12 cu. ft.; or,
$12 \times (1) = (3)$ Vol. of a solid 1 ft. l., 12 ft. w., 1 ft. th. = 12 cu. ft.; or,
$12 \times (1) = (4)$ Vol. of a solid 1 ft. l., 1 ft. w., 12 ft. th. = 12 cu. ft.

3. (1) *Multiplying the length, width, or thickness multiplies the volume; therefore,* (2) *the length, width, and thickness bear to the volume the same numerical relation as that of factors to the product.*

Question: Does Prin. 1, Art. 27, apply to example 8?

9. The interest on \$1 for 1 yr. at 1%=\$.01. Multiply by 50. (3 forms.)

Process: (1) Int. on \$1 for 1 yr. at 1%=\$.01.
50×(1)=(2) Int. on \$50 for 1 yr. at 1%=\$.50; or,
50×(1)=(3) Int. on \$1 for 50 yr. at 1%=\$.50; or,
50×(1)=(4) Int. on \$1 for 1 yr. at 50%=\$.50.

4. (1) *Multiplying the principal, time, or rate, multiplies the interest; therefore,* (2) *the principal, time, and rate bear to the interest the same numerical relation as that of factors to the product.*

Question: Does Prin. 1, Art. 27, apply to example 9?

10. The amount of wood sawed by 1 man in 1 day=1 cord. Multiply by 10. (2 forms.)

Process: (1) Amt. sawed by 1 man in 1 da.=1 c.
10×(1)=(2) Amt. sawed by 10 men in 1 da.=10 c.; or,
10×(1)=(3) Amt. sawed by 1 man in 10 da.=10 c.

5. (1) *Multiplying the number of working units or the time, multiplies the amount of work done; therefore,* (2) *the working force and the time bear to the work done the same numerical relation as that of factors to the product.*

Question: Does Prin. 1, Art. 27, apply to example 10?

11. $8 \div 4 = 2$. Multiply by 4. (2 forms.)

Process: (1) $8 \div 4 = 2$.
$4 \times (1) = (2)$ $(4 \times 8) \div 4 = 32 \div 4 = 8$; or,
$4 \times (1) = (3)$ $8 \div (4 \div 4) = 8 \div 1 = 8$. (Prin. 3, Art. 27.)

12. The length of a surface of 72 sq. ft., 8 ft. wide=9 ft. Multiply by 8. (2 forms.)

Process: (1) Length of a surface of 72 sq. ft., 8 ft. w. = 9 ft.
$8\times(1)=(2)$ Length of a surface of 576 sq. ft., 8 ft. w. = 72 ft.; or,
$8\times(1)=(3)$ Length of a surface of 72 sq. ft., 1 ft. w. = 72 ft.

NOTE.—To multiply the length, (1) multiply the area, or (2) divide the width.

Since, by principle 2 above, the *length* and *width* bear to the *area* the numerical relation of factors to a product, the numerical relation of the area and one dimension to the other dimension may be expressed as follows:

6. The area and one dimension bear to the other dimension the same numerical relation that the dividend and divisor bear to the quotient.

Question: Does Prin. 3, Art. 27, apply to example 12? Explain.

13. The width of a solid of 600 cu. ft., 15 ft. long, 5 ft. thick =8 ft. Multiply by 5. (3 forms.)

Process: (1) Width of a solid of 600 cu. ft., 15 ft. l., 5 ft. th. = 8 ft.
$5\times(1)=(2)$ Width of a solid of 3000 cu. ft., 15 ft. l., 5 ft. th. = 40 ft.; or,
$5\times(1)=(3)$ Width of a solid of 600 cu. ft., 3 ft. l., 5 ft. th. = 40 ft.; or,
$5\times(1)=(4)$ Width of a solid of 600 cu. ft., 15 ft. l., 1 ft. th. = 40 ft.

NOTE.—To multiply one dimension, (1) multiply the volume, or (2) divide either of the other dimensions.

7. The volume and two dimensions bear to the third dimension the same numerical relation that the dividend and the factors of the divisor bear to the quotient.

Question: Does Prin. 3, Art. 27, apply to example 13?

14. The principal required to gain \$36 in 6 years at 6%= \$100. Multiply by 6. (3 forms.)

Process: (1) Prin. reqd. to gain \$36 in 6 yr. at 6% = \$100.
6×(1) = (2) Prin. reqd. to gain \$216 in 6 yr. at 6% = \$600; or,
6×(1) = (3) Prin. reqd. to gain \$36 in 1 yr. at 6% = \$600; or,
6×(1) = (4) Prin. reqd. to gain \$36 in 6 yr. at 1% = \$600.

NOTE.—To multiply the principal, (1) multiply the interest, or (2) divide the time, or (3) divide the rate.

8. The interest and any two of the other elements (principal, rate, or time) bear to the third element the same numerical relation that the dividend and the factors of the divisor bear to the quotient.

Question: Does Prin. 3, Art. 27, apply to example 14?

15. The time required by 10 men to saw 10 cords of wood = 1 day. Multiply by 10. (2 forms.)

Process: (1) Time reqd. by 10 m. to saw 10 c. = 1 da.
10×(1) = (2) Time reqd. by 1 m. to saw 10 c. = 10 da.; or,
10×(1) = (3) Time reqd. by 10 m. to saw 100 c. = 10 da.

NOTE.—To multiply the time, (1) multiply the work to be done, or (2) divide the working force.

9. The work done and either element (working force or time) bear to the other element the same numerical relation that the dividend and divisor bear to the quotient.

EXERCISE XLIII.

1. 17 − 4 + 6 = 19. Multiply by 10.
2. 1 bushel = 4 pecks. Multiply by 80.
3. 3 quarts = 6 pints. Multiply by 12.
4. 39.37 inches = 1 meter. Multiply by 200.
5. 1 ton = 20 hundredweight. Multiply by 6.
6. \$50 − 4 × A's money = \$30. Multiply by 13.
7. Cost of 1 pound = 8¢. Multiply by 14. (2 forms.)
8. Value of 1 book = 75¢. Multiply by 11. (2 forms.)

9. Wt. of 12 loads = 150000 pounds. Multiply by 8. (2 forms.)

10. Price of 15 shares = \$750. Multiply by 20. (2 forms.)

11. $8 \times 9 = 72$. Multiply by 7. (2 forms.)

12. Area of a surface 1 ft. long, 1 ft. wide = 1 sq. ft. Multiply by 7. (2 forms.)

13. Volume of a solid 1 ft. long, 1 ft. wide, 1 ft. thick = 1 cu. ft. Multiply by 18. (3 forms.)

14. $72 \div 8 = 9$. Multiply by 4. (2 forms.)

15. Width of a surface of 30 sq. ft., 15 ft. long = 2 ft. Multiply by 15. (2 forms.)

16. Length of a surface of 30 sq. ft., 2 ft. wide = 15 ft. Multiply by 5. (2 forms.)

17. Time required for \$500 to gain \$80 at 8% = 2 yr. Multiply by 3. (3 forms.)

18. Length of a wall 12 ft. high, 2 ft. thick, built by 6 men in 8 days = 600 ft. Multiply by 2. (4 forms.)

19. Principal required to produce \$80 at 8% in 2 years = \$500. Multiply by 12. (3 forms.)

20. Rate required for \$500 to produce \$80 in 2 years = 8%. Multiply by 7. (3 forms.)

21. Thickness of a solid of 30 cu. ft., 6 ft. long, 5 ft. wide = 1 ft. Multiply by 10. (3 forms.)

62. To Divide an Equation.—The *Third Application of Division* is employed in dividing equations. An equation is divided by taking the same part of each of its members.

GENERAL PRINCIPLES: *1. Equals divided by the same number give equal quotients.*

2. Dividing each term of a member of an equation by a number divides the member by that number.

NOTE.—In addition to these general principles, all but one of the special principles given in Article 61 will be of service in the following examples and exercise.

EXAMPLES.

1. 21 bushels=84 pecks. Divide by 7.

NOTE.—Remember, that to divide by 7 means to take $\frac{1}{7}$ of each member.

Process: (1) 21 bu. = 84 pk.
$\frac{1}{7}$ of (1) = (2) 3 bu. = 12 pk.

2. Cost of 9 hats=\$36. Divide by 9. (2 forms.)

Process: (1) Cost of 9 hats = \$36.
$\frac{1}{9}$ of (1) = (2) $\frac{1}{9}$ of cost of 9 hats = \$4; or,
$\frac{1}{9}$ of (1) = (3) Cost of 1 hat = \$4.

NOTE.—Observe that either equation (2) or (3) is the correct result, and that $\frac{1}{9}$ *of cost of 9 hats = cost of 1 hat.*

3. $25-15+30=40$ Divide by 5.

Process: (1) $25-15+30=40$.
$\frac{1}{5}$ of (1) = (2) $5-3+6=8$. (Prin. 2.)

4. 20×my age−120 years=8×John's age. Divide by 4.

Process: (1) 20×my age − 120 yr. = 8×John's age.
$\frac{1}{4}$ of (1) = (2) 5×my age − 30 yr. = 2×John's age.

5. $12\times 18=216$. Divide by 6. (2 forms.)

Process: (1) $12\times 18=216$.
$\frac{1}{6}$ of (1) = (2) $\frac{12}{6}\times 18=2\times 18=36$; or,
$\frac{1}{6}$ of (1) = (3) $12\times\frac{18}{6}=12\times 3=36$. (Prin. 2, Art. 27.)

6. Area of a surface 15 feet long and 12 feet wide=180 square feet. Divide by 3. (2 forms.)

Process: (1) Area of a surface 15 ft. l., 12 ft. w. = 180 sq. ft.
$\frac{1}{3}$ of (1) = (2) Area of a surface 5 ft. l., 12 ft. w. = 60 sq. ft.; or,
$\frac{1}{3}$ of (1) = (3) Area of a surface 15 ft. l., 4 ft. w. = 60 sq. ft.

NOTE.—Show how Prin. 2, p. 85, and Prin. 2, Art. 27, apply to example 6.

7. The volume of a solid 18 ft. long, 12 ft. wide, and 6 ft. thick=1296 cu. ft. Divide by 6. (3 forms.)

Process: (1) Vol. of a solid 18 ft. l., 12 ft. w., 6 ft. th. = 1296 cu. ft.
$\frac{1}{6}$ of (1) = (2) Vol. of a solid 3 ft. l., 12 ft. w., 6 ft. th. = 216 cu. ft.; or,
$\frac{1}{6}$ of (1) = (3) Vol. of a solid 18 ft. l., 2 ft. w., 6 ft. th. = 216 cu. ft.; or,
$\frac{1}{6}$ of (1) = (4) Vol. of a solid 18 ft. l., 12 ft. w., 1 ft. th. = 216 cu. ft.

NOTE.—Show how Prin. 3 (2), Art. 61, and Prin. 2, Art. 27, apply to example 7.

8. $72 \div 12 = 6$. Divide by 2. (2 forms.)

Process: (1) $72 \div 12 = 6$.
$\frac{1}{2}$ of (1) = (2) $\frac{72}{2} \div 12 = 36 \div 12 = 3$; or,
$\frac{1}{2}$ of (1) = (3) $72 \div (2 \times 12) = 72 \div 24 = 3$. (Prin. 4, Art. 27.)

9. Rate required for \$500 to produce \$750 in 15 years = 10%. Divide by 5. (3 forms.)

Process: (1) Rate reqd. for \$500 to produce \$750 in 15 yr. = 10%.
$\frac{1}{5}$ of (1) = (2) Rate reqd. for \$2500 to produce \$750 in 15 yr. = 2%; or,
$\frac{1}{5}$ of (1) = (3) Rate reqd. for \$500 to produce \$150 in 15 yr. = 2%; or,
$\frac{1}{5}$ of (1) = (4) Rate reqd. for \$500 to produce \$750 in 75 yr. = 2%.

NOTE.—Show how Prin. 8, Art. 61, and Prin. 4, Art. 27, apply to example 9.

10. The time required by 10 men to saw 120 cords of wood = 10 days. Divide by 5. (2 forms.)

Process: (1) Time reqd. for 10 m. to saw 120 c. = 10 da.
$\frac{1}{5}$ of (1) = (2) Time reqd. for 50 m. to saw 120 c. = 2 da.; or,
$\frac{1}{5}$ of (1) = (3) Time reqd. for 10 m. to saw 24 c. = 2 da.

NOTE.—Show how Prin. 9, Art. 61, and Prin. 4, Art. 27, apply to example 10.

11. Thickness of a solid of 480 cu. ft., 10 ft. long, 8 ft. wide = 6 ft. Divide by 2. (3 forms.)

Process: (1) Thickness of a solid of 480 cu. ft., 10 ft. l., 8 ft. w. = 6 ft.
$\frac{1}{2}$ of (1) = (2) Thickness of a solid of 480 cu. ft., 10 ft. l., 16 ft. w. = 3 ft.; or,
$\frac{1}{2}$ of (1) = (3) Thickness of a solid of 240 cu. ft., 10 ft. l., 8 ft. w. = 3 ft.; or,
$\frac{1}{2}$ of (1) = (4) Thickness of a solid of 480 cu. ft., 20 ft. l., 8 ft. w. = 3 ft.

Question: Why increase the width in (2)? the length in (4)?

NOTE.—Show how Prin. 7, Art. 61, and Prin. 4, Art. 27, apply to example 11.

EXERCISE XLIV.

1. 18−12+24=30. Divide by 6.

2. 420 hundredweight=21 tons. Divide by 7.

3. \$560−40% of cost=\$280. Divide by 40.

4. 18×30=540. Divide by 6. (2 forms.)

5. 4×12×10=480. Divide by 2. (3 forms.)

6. 50% of 75% of a number=250. Divide by 25. (2 forms.)

7. Weight of 20 barrels=5000 pounds. Divide by 10. (2 forms.)

8. Cost of 30 oranges=40¢. Divide by 30. (2 forms.)

9. The volume of a solid 12 ft. long, 4 ft. high, 8 ft. thick =384 cu. ft. Divide by 4. (3 forms.)

10. Interest on \$6000 at 8% for $1\frac{1}{3}$ yr.=\$640. Divide by 8. (3 forms.)

11. 24÷8=3. Divide by 3. (2 forms.)

12. Time required by 14 women to make 7 dresses=$3\frac{1}{2}$ days. Divide by $3\frac{1}{2}$. (2 forms.)

13. Width of a surface of 108 sq. ft., 12 ft. long=9 ft. Divide by 9. (2 forms.)

14. Length of a wall 12 ft. high, 2 ft. thick, built by 6 men in 12 days=900 ft. Divide by 6. (4 forms.)

15. Principal required to produce \$640 in $1\frac{1}{3}$ years at 8%=\$6000. Divide by 64. (3 forms.)

16. Rate required for \$6000 to produce \$640 in $1\frac{1}{3}$ years=8%. Divide by 4. (3 forms.)

17. Time required for \$6000 to produce \$640 at 8%=$1\frac{1}{3}$ years. Divide by 4. (3 forms.)

2. RATIO.

63. Definitions.—A *ratio* is an expression of the relative magnitude of one number as compared with another of the same kind taken as the standard. The number compared is the *antecedent;* the one taken as the standard of comparison is the *consequent*. These numbers are also the *terms* of the ratio, and together form a *couplet*.

$$15 \text{ yd.} : 5 \text{ yd.} = 3$$

is read, "The ratio of 15 yd. to 5 yd. equals 3," and means that 15 yd. is 3 times 5 yd.

$$\$8 : \$12 = \tfrac{2}{3}$$

means that $\$8$ is $\frac{2}{3}$ of $\$12$. Observe that a ratio is always *abstract*. 15 yd. is *3* times 5 yd., but not *3 yd.* times 5 yd.; $\$8$ is $\frac{2}{3}$ of $\$12$, but not $\$\frac{2}{3}$ of $\$12$.

In arithmetic we have the *indicated ratio*, and the *numerical ratio*. As,

In $\$32 : \$16 = 2$

$\$32 : \$16$ (indicated ratio).

2 (numerical ratio).

For every *indicated ratio* whose terms are *concrete* there may be written a corresponding *indicated ratio* whose terms are *abstract*. As,

70 bu. :	14 bu.	(Terms concrete.)
70 :	14	(Terms abstract.)

NOTE.—The respective terms in the two ratios are numerically equal; the ratios are absolutely equal.

PRINCIPLE: *The numerical ratio between any two similar concrete numbers is the same as the numerical ratio between their corresponding abstract numbers.*

64. Writing the Corresponding Abstract Terms of Ratios.—In solving ratios and proportions, it is best to deal with abstract terms. The following principles will be of service in determining the corresponding abstract terms of ratios whose terms are concrete.

NOTE.—These principles are found in Article 61, but are not grouped there as they are here.

I. PRINCIPLES RELATING TO SURFACES: (1) *The dimensions (length and width) bear to the area the same numerical relation as that of factors to the product.* (2) *The area and one dimension bear to the other dimension the same numerical relation that the dividend and divisor bear to the quotient.*

II. PRINCIPLES RELATING TO SOLIDS: (1) *The dimensions (length, width, and thickness) bear to the volume the same numerical relation as that of factors to the product.* (2) *The volume and two dimensions bear to the third dimension the same numerical relation that the dividend and the factors of the divisor bear to the quotient.*

III. PRINCIPLES RELATING TO INTEREST: (1) *The principal, time, and rate bear to the interest the same numerical relation as that of factors to the product.* (2) *The interest and any two of the elements (principal, rate, or time) bear to the third element the same numerical relation that the dividend and the factors of the divisor bear to the quotient.*

IV. PRINCIPLES RELATING TO WORK: (1) *The working force and time bear to the work done the same numerical relation as that of factors to the product.* (2) *The work done and either of the elements (working force or time) bear to the other element the same numerical relation that the dividend and divisor bear to the quotient.*

EXAMPLES.

The following are examples of writing corresponding abstract terms of ratios:

1. 70 bushels : 14 bushels.

Abstract: 70 : 14.

2. 5×20 men : 4×8 men.

Abstract: $5 \times 20 : 4 \times 8$.

3. $\frac{1}{5}$ of cost of 32 books : $\frac{3}{4}$ of cost of 21 books.

Abstract: $\frac{1}{5}$ of 32 : $\frac{3}{4}$ of 21.

4. $\left\{\begin{matrix}\text{Area of a surface}\\ \text{12 ft. long, 9 ft wide}\end{matrix}\right\} : \left\{\begin{matrix}\text{area of a surface}\\ \text{9 ft. long, 8 ft. wide}\end{matrix}\right\}.$

Abstract: $12\times 9:9\times 8$. (Prin. I-(1).)

5. $\left\{\begin{matrix}\text{Length of a solid of}\\ \text{72 cu. ft., 4 ft. wide,}\\ \text{2 ft. thick}\end{matrix}\right\} : \left\{\begin{matrix}\text{length of a solid of}\\ \text{90 cu. ft., 5 ft. wide,}\\ \text{3 ft. thick}\end{matrix}\right\}.$

Abstract: $\frac{72}{4\times 2}:\frac{90}{5\times 3}$. (Prin. II-(2).)

6. $\left\{\begin{matrix}\text{Interest on \$500}\\ \text{at 6\% for } 3\tfrac{1}{2}\text{ yr.}\end{matrix}\right\} : \left\{\begin{matrix}\text{interest on \$400}\\ \text{at 8\% for 5 yr.}\end{matrix}\right\}.$

Abstract: $500\times 6\times 3\frac{1}{2}:400\times 8\times 5$. (Prin. III-(1).)

7. $\left\{\begin{matrix}\text{Rate required for}\\ \text{\$500 to gain \$90}\\ \text{in 3 years}\end{matrix}\right\} : \left\{\begin{matrix}\text{rate required for}\\ \text{\$400 to gain \$160}\\ \text{in 5 years}\end{matrix}\right\}.$

Abstract: $\frac{90}{500\times 3}:\frac{160}{400\times 5}$. (Prin. III-(2).)

8. $\left\{\begin{matrix}\text{Work done by}\\ \text{15 men in 12 days}\end{matrix}\right\} : \left\{\begin{matrix}\text{work done by}\\ \text{11 men in 8 days}\end{matrix}\right\}.$

Abstract: $15\times 12:11\times 8$. (Prin. IV-(1).)

9. $\left\{\begin{matrix}\text{Number of men re-}\\ \text{quired to build 180 rd.}\\ \text{of fence in 12 days}\end{matrix}\right\} : \left\{\begin{matrix}\text{number of men re-}\\ \text{quired to build 88 rd.}\\ \text{of fence in 8 days}\end{matrix}\right\}.$

Abstract: $\frac{180}{12}:\frac{88}{8}$. (Prin. IV-(2).)

EXERCISE XLV.

Write the corresponding ratios with abstract terms:

1. 25 lb. : 15 lb.

2. 8×9 books : 7×12 books.

3. Cost of 10 shares : cost of 15 shares.

4. Value of 20 loads : value of 14 loads.

5. $7\times$my age : $12\times$my age.

6. $9\times$cost of 10 hats : $12\times$cost of 5 hats.

7.	Area of a surface 10 rd. long, 8 rd. wide	:	area of a surface 17 rd. long, 12 rd. wide	.
8.	Length of a surface of 160 sq. ft., 8 ft. wide	:	length of a surface of 360 sq. ft., 18 ft. wide	.
9.	Width of a surface of 380 sq. rd., 20 rd. long	:	width of a surface of 540 sq. rd., 30 rd. long	.
10.	Volume of a solid 20 ft. long, 8 ft. high, 2 ft. thick	:	volume of a solid 30 ft. long, 10 ft. high, $1\frac{1}{2}$ ft. thick	.
11.	Length of a solid of 140 cu. yd., 7 yd. high, 2 yd. thick	:	length of a solid of 120 cu. yd., 5 yd. high, 3 yd. thick	.
12.	Width of a solid of 120 cu. yd., 8 yd. long, 3 yd. thick	:	width of a solid of 140 cu. ft., 10 ft. long, 2 ft. thick	.
13.	Thickness of a solid of 108 cu. ft., 12 ft. long, 9 ft. wide	:	thickness of a solid of 120 cu. ft., 10 ft. long, 8 ft. wide	.
14.	Interest on \$500 for 6 yr. at 10%	:	interest on \$750 for 3 yr. at 8%	.
15.	Time reqd. for \$600 to gain \$75 at 6%	:	time reqd. for \$480 to gain \$60 at 10%	.
16.	Rate reqd. for \$540 to bear \$86.40 in 2 yr.	:	rate reqd. for \$620 to bear \$248 in 4 yr.	.
17.	Work done by 12 men in 25 days	:	work done by 16 men in 20 days	.
18.	Time reqd. by 6 men to build 40 rd. of fence	:	time reqd. by 15 men to build 120 rd. of fence	.
19.	Number of men reqd. to build 140 rd. in 18 days	:	number of men reqd. to build 320 rd. in 15 days	.
20.	Principal reqd. to produce \$112 in $2\frac{1}{2}$ yr. at 7%	:	principal reqd. to produce \$120 in $3\frac{1}{3}$ yr. at 5%	.

65. The Process of Finding the Numerical Ratio.—The process employed in determining the numerical ratio between two numbers is *division*. (See Second Application, Art. 16.) The antecedent becomes the dividend; the consequent, the divisor; and the numerical ratio, the quotient.

PRINCIPLES: *1. The antecedent divided by the consequent equals the numerical ratio.*

2. When the terms are abstract, the antecedent divided by the numerical ratio equals the consequent.

3. The product of the consequent multiplied by the numerical ratio equals the antecedent.

EXAMPLES.

1. The antecedent is \$20, the consequent \$5. Find the numerical ratio.

Statement: \$20 : \$5 = ( )?

Process: 20 ÷ 5 = 4.

NOTE.—It is recommended that, in the *process*, the corresponding abstract terms be always used; then, no trouble will arise in dealing with complicated expressions, such as are found below.

The *statement* is made for the purpose of showing clearly the relation of the *given parts* to the *required part*.

2. 12 is the ratio of 72 days to what?

Statement: 72 da. : ( ) = 12?

Process: 72 ÷ 12 = 6. (Prin. 2.)

∴ 6 days is the consequent.

Question: Why is the consequent *days*?

3. 10 is the ratio of what number to 13 men?

Statement: ( ) : 13 men = 10.

Process: 10 × 13 = 130. (Prin. 3.)

∴ 130 men is the antecedent.

4. Find the ratio of the work of 6 men for 15 days to the work of 9 men for 2 days.

$$\textit{Statement:}\ \left\{\begin{matrix}\text{Work of 6 men}\\ \text{for 15 days}\end{matrix}\right\} : \left\{\begin{matrix}\text{work of 9 men}\\ \text{for 2 days}\end{matrix}\right\} = (\ )?$$

$$\textit{Abstract:}\ 6\times 15 : 9\times 2 = (\ )?$$

$$\textit{Process:}\ \frac{6\times 15}{9\times 2} = 5.$$

5. One block of marble is 10 ft. long, 5 ft. wide, 2 ft. thick; another is 15 ft. long, 4 ft. wide, 5 ft. thick. The volume of the first bears what ratio to the volume of the second?

$$\textit{Statement:}\ (1)\ \left\{\begin{matrix}\text{Volume}\\ \text{10 ft. l.,}\\ \text{5 ft. w.,}\\ \text{2 ft. th.}\end{matrix}\right\} : \left\{\begin{matrix}\text{volume}\\ \text{15 ft. l.,}\\ \text{4 ft. w.,}\\ \text{5 ft. th.}\end{matrix}\right\} = (\ )?$$

$$(2)\ \left\{\begin{matrix}10\times\\ 5\times\\ 2\end{matrix}\right\} : \left\{\begin{matrix}15\times\\ 4\times\\ 5\end{matrix}\right\} = (\ )?$$

$$\textit{Process:}\ \frac{10\times 5\times 2}{15\times 4\times 5} = \tfrac{1}{3}.$$

NOTE.—Employ cancellation wherever convenient.

6. A tract of land 50 rd. long, 30 rd. wide, is $1\frac{1}{3}$ times as large as another tract, 45 rd. long. How wide is the second tract?

$$\textit{Statement:}\ (1)\ \left\{\begin{matrix}\text{Tract}\\ \text{50 rd. l.,}\\ \text{30 rd. w.}\end{matrix}\right\} : \left\{\begin{matrix}\text{tract}\\ \text{45 rd. l.,}\\ \text{( ) rd. w.}\end{matrix}\right\} = 1\tfrac{1}{3}?$$

$$(2)\ \left\{\begin{matrix}50\times\\ 30\end{matrix}\right\} : \left\{\begin{matrix}45\times\\ (\ )\end{matrix}\right\} = 1\tfrac{1}{3}?$$

$$\textit{Process:}\ \frac{50\times 30}{45\times 1\frac{1}{3}} = 25.$$

$\therefore$ the required width is 25 rd.

7. What is the ratio of the principal required to produce \$216 interest in 5 yr. at 6% to the principal required to produce \$192 in 4 yr. at 8%?

$$\textit{Statement:}\ (1)\ \left\{\begin{matrix}\text{Prin. reqd. to}\\ \text{produce \$216 in}\\ \text{5 yr. at 6\%}\end{matrix}\right\} : \left\{\begin{matrix}\text{prin. reqd. to}\\ \text{produce \$192 in}\\ \text{4 yr. at 8\%}\end{matrix}\right\} = (\ )?$$

$$(2)\ \frac{216}{5\times 6} : \frac{192}{4\times 8} = (\ )?$$

Process: $\frac{216 \times 4 \times 8}{5 \times 6 \times 192} = \frac{6}{5} = 1\frac{1}{5}$.

8. Find the ratio of the length of a surface of 144 sq. ft., 12 ft. wide, to the length of a surface of 180 sq. ft., 10 ft. wide.

Statement: (1) $\left\{\begin{array}{l}\text{Length of a surface}\\ \text{of 144 sq. ft., 12 ft.}\\ \text{wide}\end{array}\right\} : \left\{\begin{array}{l}\text{length of a surface}\\ \text{of 180 sq. ft., 10 ft.}\\ \text{wide}\end{array}\right\} = (\ )?$

(2) $\frac{144}{12} : \frac{180}{10} = (\ )?$

Process: $\frac{144 \times 10}{12 \times 180} = \frac{2}{3}$.

EXERCISE XLVI.

Find the required part in each of the following:

FIND THE RATIO.		FIND THE CONSEQUENT.		FIND THE ANTECEDENT.	
Antecedent.	*Consequent.*	*Antecedent.*	*Ratio.*	*Consequent.*	*Ratio.*
1. 56 ft.	8 ft.	*7.* \$64.	11.	*13.* 342 in.	$\frac{1}{2}$.
2. 540 yd.	18 yd.	*8.* 16 da.	$\frac{1}{3}$.	*14.* 34 mi.	$\frac{5}{9}$.
3. 2760 men.	60 men.	*9.* 98 sq. ft.	10.	*15.* 72 yr.	13.
4. \$$\frac{1}{2}$.	\$20.	*10.* \$7000.	140.	*16.* .003 oz.	440.
5. 34 bu.	16 bu.	*11.* $\frac{3}{8}$ hr.	72.	*17.* 64 cords.	9.
6. 12 hr.	54 hr.	*12.* 24 gal.	$1\frac{5}{7}$.	*18.* $\frac{345}{926}$ meters.	1852.

Find the numerical ratios in Exercise XLV.

3. PROPORTION.

66. Definitions.—A proportion is the expression of equality between two indicated ratios. Every proportion, therefore, has 4 terms, which are numbered in order from left to right. The first and fourth terms are the *extremes;* the second and third terms, the *means.* The *double colon* (: :) is usually (but not always) used for the sign of equality between the ratios.

8 bu. : 3 bu. : : \$40 : \$15,

is read, "8 bu. is to 3 bu. as \$40 is to \$15," and means that the ratio of 8 bu. to 3 bu. is equal to the ratio of \$40 to \$15.

67. Denominations of Terms.—(1) All the terms of a proportion may be concrete. As,

$$\$25 : \$12 :: 100 \text{ men} : 48 \text{ men}.$$

Remember, that the two terms of an indicated ratio must be similar; but terms in different ratios need not be similar, even though those ratios are in the same proportion.

(2) One couplet may be concrete and the other abstract. As,

$$\$25 : \$12 :: 100 : 48.$$

(3) All the terms may be abstract. As,

$$25 : 12 :: 100 : 48.$$

By dropping the denominations in a proportion, all or a part of whose terms are concrete, a new proportion is formed whose terms are abstract but are numerically equal to those of the first proportion. Thus,

(1) 30 yr. : 12 yr. :: 90 bu. : 36 bu. (Terms concrete.)
(2) 30 : 12 :: 90 : 36. (Terms abstract.)

68. Solving a Proportion with Abstract Terms.—Solving a proportion is the process of finding any one of its terms when the other three are known.

PRINCIPLES: *1. In any proportion whose terms are abstract, the product of the extremes is equal to the product of the means.*

Thus, (1) $7 : 15 :: 21 : 45$.
Then, $7 \times 45 = 15 \times 21$.
(2) $8 : 3 :: 32 : 12$.
Then, $8 \times 12 = 3 \times 32$.

2. Either extreme is equal to the product of the means divided by the other extreme.

Thus, $20 : 34 :: 30 : 51$.
Then, $\frac{34 \times 30}{20} = 51$, and $\frac{34 \times 30}{51} = 20$.

3. Either mean is equal to the product of the extremes divided by the other mean.

Thus, 36 : 14 : : 18 : 7.

Then, $\frac{36\times7}{14}=18$, and $\frac{36\times7}{18}=14$.

EXAMPLES.

1. Find the missing term: 12 : 25 : : ( ) : 125 ?

Process : $\frac{12\times125}{25}=60.$

NOTE.—Use cancellation where convenient.

2. Find the missing term: ( ) : 32 : : 60 : 8 ?

Process : $\frac{32\times60}{8}=240.$

3. Find the missing term: 13 : 12 : : 78 : ( ) ?

Process : $\frac{12\times78}{13}=72.$

4. Find the missing term: 9 : ( ) : : 72 : 4 ?

Process : $\frac{9\times4}{72}=\frac{1}{2}.$

5. Find the missing factor: (12×6) : 35 : : ( )×8 : 70 ?

I. *Process :* (1) $\frac{12\times6\times70}{35}=144.$

(2) $144\div8=18.$

Or, II. *Process :* $\frac{12\times6\times70}{35\times8}=18.$

NOTE.—The plan of the first process is to neglect the *8* and solve for the entire third term (144); then, dividing by *8* will give the required factor. The plan of the second process is this: Since the product of all the factors of the extremes equals the product of all the factors of the means, the product of all the factors of the extremes divided by the product of all but one of the factors of the means, will give that one.

6. $\dfrac{120 \times 100}{(\ \ ) \times 6} : \dfrac{108 \times 100}{720 \times 5} :: 12 : 9$?

Process : $\dfrac{108 \times 100 \times 12 \times 6}{720 \times 5 \times 120 \times 100 \times 9} = \dfrac{1}{500}$.

$\therefore$ the missing factor is 500.

NOTE.—Where the *fractions* are to be multiplied together, their numerators are placed above and their denominators are placed below the line (why ?); when used as divisors, their denominators are placed above and their numerators below the line (why ?). Special attention should be given to the fact, that if the required factor is in a denominator, it will appear in the result as a denominator with 1 for its numerator.

7. $\dfrac{128}{16} : \dfrac{72}{(\ \ )} :: 25 : 45$?

Process : $\dfrac{128 \times 45}{16 \times 72 \times 25} = \dfrac{1}{5}$.

$\therefore$ the required factor is 5 (in the denominator).

EXERCISE XLVII.

1. $12 : 3 :: (\ \ ) : 4$?
2. $12 : 3 :: 4 : (\ \ )$?
3. $(\ \ ) : 72 :: 1\frac{1}{2} : 3$?
4. $43 : (\ \ ) :: 20 : 600$?
5. $221 : 13 :: (\ \ ) : 21$?
6. $(\ \ ) : 1\frac{1}{2} :: 360 : 720$?
7. $\frac{2}{21} : (\ \ ) :: \frac{3}{7} : 9$?
8. $9\frac{3}{5} : 4 :: 360 : (\ \ )$.
9. $\dfrac{125}{46 \times (\ \ )} : \dfrac{1}{7} :: \dfrac{1}{36} : \dfrac{23}{175}$?
10. $\dfrac{8 \times 5}{19 \times 7} : \dfrac{25 \times 24}{9 \times (\ \ )} :: 12 : 72$?
11. $27 : \dfrac{12 \times 6}{14 \times 28} :: \dfrac{108}{19} : \dfrac{(\ \ )}{49}$?
12. $\dfrac{162}{540 \times 5} : \dfrac{210}{700 \times 6} :: (\ \ ) : 5$?

69. Solving a Proportion with Concrete Terms.—In any proportion whose terms are concrete, a missing part may be found as follows:

First, form and solve the corresponding proportion whose terms are abstract ; then, determine from the given proportion by inspection the denomination of the required term.

EXAMPLES.

Find the missing part:

1. 7 da. : 12 da. : : ( ) mo. : 72 mo.?

Abstract terms: 7 : 12 : : ( ) : 72 ?

Process: $\frac{7\times 72}{12}=42.$

$\therefore$ 42 mo. is the required term.

2. $\left\{\begin{array}{l}\text{Area of a surface}\\ \text{40 rd. long, 20}\\ \text{rd. wide}\end{array}\right\} : \left\{\begin{array}{l}\text{area of a surface}\\ \text{80 rd. long, ( )}\\ \text{rd. wide}\end{array}\right\}$: : 5 A. : 30 A.?

Abstract terms: $20\times 40 : 80\times(\ ) :: 5 : 30$?

Process: $\frac{20\times 40\times 30}{80\times 5}=60.$

$\therefore$ 60 rd. is the required part.

3. $\left\{\begin{array}{c}\text{800 reams of}\\ \text{16 pages to}\\ \text{the sheet}\end{array}\right\} : \left\{\begin{array}{c}\text{( ) reams of}\\ \text{24 pages to}\\ \text{the sheet}\end{array}\right\} :: \left\{\begin{array}{c}\text{38400 vol.}\\ \text{of 160 pp.}\\ \text{each}\end{array}\right\} : \left\{\begin{array}{c}\text{27000 vol.}\\ \text{of 320 pp.}\\ \text{each}\end{array}\right\}$?

Abstract terms: $\left\{\begin{array}{c}800\\ \times 16\end{array}\right\} : \left\{\begin{array}{c}(\ )\\ \times 24\end{array}\right\} :: \left\{\begin{array}{c}38400\\ \times 160\end{array}\right\} : \left\{\begin{array}{c}27000\\ \times 320\end{array}\right\}$?

Process: $\frac{800\times 16\times 27000\times 320}{24\times 38400\times 160}=750.$

$\therefore$ 750 reams is the required term.

4. $\left\{\begin{array}{c}\text{Body of soldiers}\\ \text{( ) men long, 60}\\ \text{men wide}\end{array}\right\} : \left\{\begin{array}{c}\text{body of soldiers}\\ \text{100 men long, 75}\\ \text{men wide}\end{array}\right\}$: : No. of men : itself ?

Abstract terms: $(\ )\times 60 : 100\times 75 ::$ No. : No. ?

Process: $\frac{100\times 75\times \text{No.}}{60\times \text{No.}}=125.$

$\therefore$ 125 men is the required part.

NOTE.—The "No." being in both numerator and denominator, cancels.

5. $\left\{\begin{array}{c}\text{Time reqd. by}\\ \text{( ) men to build}\\ \text{102 rd. of fence}\end{array}\right\} : \left\{\begin{array}{c}\text{time reqd. by}\\ \text{8 men to build}\\ \text{680 rd. of fence}\end{array}\right\}$: : 34 : 170 ?

Abstract terms: $\frac{102}{(\ )} : \frac{680}{8} :: 34 : 170$?

$$\textit{Process}: \frac{680\times 34}{8\times 102\times 170}=\tfrac{1}{6}.$$

$\therefore$ **6 men is the required part.**

Note.—Do not forget to use the reciprocal of the result, when the required term is a factor of the denominator.

6. $\left\{\begin{array}{l}\text{Principal reqd. to}\\ \text{gain \$112 in ( ) yr.}\\ \text{at }7\%\end{array}\right\} : \left\{\begin{array}{l}\text{principal reqd. to}\\ \text{gain \$120 in }3\tfrac{1}{3}\text{ yr.}\\ \text{at }5\%\end{array}\right\} :: 8:9\ ?$

$$\textit{Abstract terms}: \frac{112}{(\ )\times 7}:\frac{120}{3\frac{1}{3}\times 5}::8:9\ ?$$

$$\textit{Process}: \frac{120\times 8\times 7\times 3}{112\times 5\times 9\times 10}=\tfrac{2}{5}.$$

$\therefore$ **$\frac{5}{2}$ or $2\frac{1}{2}$ yr. is the reqd. part.**

EXERCISE XLVIII.

Find the missing part in each of the following:

1. \$120 : ( ) : : 48 cords : 20 cords ?

Note.—The 2d term must be \$'s; why ?

2. \$3.50 : \$50 : : 8 yd. : ( ) ?

Note.—The 4th term must be yards; why ?

3. 44 in. : 18 in. : : ( ) : 900 in. ?

4. $\left\{\begin{array}{c}\text{Vol. of a solid}\\ \text{20 ft. long, 8 ft.}\\ \text{wide, ( ) ft. thick}\end{array}\right\} : \left\{\begin{array}{c}\text{vol. of a solid}\\ \text{30 ft. long, 10 ft.}\\ \text{wide, }1\tfrac{1}{2}\text{ ft. thick}\end{array}\right\} :: 32:45\ ?$

5. $\left\{\begin{array}{c}\text{Length of a solid}\\ \text{140 cu. yd., 7 yd.}\\ \text{high, 2 yd. thick}\end{array}\right\} : \left\{\begin{array}{c}\text{length of a solid}\\ \text{( ) cu. yd., 5 yd.}\\ \text{high, 3 yd. thick}\end{array}\right\} :: 20:16\ ?$

6. $\left\{\begin{array}{c}\text{Width of a solid}\\ \text{150 cu. ft., 9 ft.}\\ \text{high, 2 ft. thick}\end{array}\right\} : \left\{\begin{array}{c}\text{width of a solid}\\ \text{180 cu. ft., ( ) ft.}\\ \text{high, 3 ft. thick}\end{array}\right\} :: 25:18\ ?$

7. $\left\{\begin{array}{c}\text{Thickness of a solid}\\ \text{162 cu. ft., 12 ft.}\\ \text{long, 9 ft. high}\end{array}\right\} : \left\{\begin{array}{c}\text{thickness of a solid}\\ \text{216 cu. ft., 9 ft.}\\ \text{long, 8 ft. high}\end{array}\right\} :: (\ ):2\ ?$

8. $\left\{\begin{array}{l}\text{Interest on \$500}\\ \text{for 6 yr. at 10\%}\end{array}\right\} : \left\{\begin{array}{l}\text{interest on \$720}\\ \text{for ( ) yr. at 8\%}\end{array}\right\} :: 5 : 3$?

9. $\left\{\begin{array}{l}\text{Time reqd. by 6}\\ \text{men to build 40}\\ \text{rd. of fence}\end{array}\right\} : \left\{\begin{array}{l}\text{time reqd. by 15}\\ \text{men to build 120}\\ \text{rd. of fence}\end{array}\right\} :: 5 : (\ \ )$?

10. $\left\{\begin{array}{c}\text{Number of men reqd.}\\ \text{to build 320 rd. of}\\ \text{fence in 15 days}\end{array}\right\} : \left\{\begin{array}{c}\text{number of men reqd.}\\ \text{to build 140 rd. of}\\ \text{fence in ( ) days}\end{array}\right\} :: 96 : 35$?

B. PROBLEMS OF ONE BASIS—NATURE AND CLASSIFICATION.

70. Nature.—Typically, a problem consists of two parts: (1) A question, proposed for solution; and (2) a statement of a certain condition or relation, from which that solution may be determined. These parts are called **Question** and **Basis.** Thus,

If 6 books cost \$2.40, what cost 10 books ?

Parts : (1) What cost 10 books ? (Question.)
(2) 6 books cost \$2.40. (Basis.)

NOTE.—The above description refers only to such problems as require but one unknown part or term to be found. One *basis* only is needed for the solution of such a problem. In Part III, particular attention will be given to problems of two or more bases.

Comparatively few problems are stated in the form of the type; yet, all problems have these two essential parts, *expressed or implied.*

The following are some of the forms not typical:

1. Find $\frac{3}{4}$ of 200 yards.
2. 150 ounces is $\frac{3}{4}$ of what ?
3. Reduce 40000 lb. to tons.
4. \$210 is what part of \$450 ?
5. Find the area of a floor 20 ft. long and 15 ft. wide.
6. What cost 6 doz. eggs at 20¢ per doz.?

These may all be reduced to the form of the type. Thus,

1. Since all ($\frac{4}{4}$) of 200 yd. is 200 yd., what is $\frac{3}{4}$ of it?

2. If $\frac{3}{4}$ of a number is 150 ounces, what is the number?

3. Since 2000 lb. make 1 ton, how many tons in 40000 lb.?

4. Since \$450 is all of \$450, \$210 is what part of it?

5. Since the area of a surface 1 ft. long, 1 ft. wide is 1 sq. ft., what is the area of a floor 20 ft. long and 15 ft. wide?

6. If 1 doz. eggs cost 20¢, what cost 6 doz.?

71. Stating the Parts of a Problem.—Each part of a problem may be stated in the form of an **equation.** Take for example the parts—

(1) What cost 10 books? (Question.)
(2) 6 books cost \$2.40. (Basis.)

The **question** may be stated as an equation between *cost of 10 books* and the *required term.* Thus,

Cost of 10 books = ( ) ?

In like manner, the **basis** may be stated as an equation between *cost of 6 books* and *\$2.40.* Thus,

(1) Cost of 6 books = \$2.40; or,
(2) \$2.40 = cost of 6 books.

Problems often involve the relation of more than two numbers. The *basis* may have as many forms as it has different numbers related. The following is an example:

If 15 men harvest 200 acres of grain in 10 days, in how many days can 5 men harvest 30 acres?

The relation here is among *men* (working force), *time*, and *work done;* and its three forms are:

(1) Amt. harvested by 15 men in 10 days = 200 A.; or,
(2) Time reqd. for 15 men to harvest 200 A. = 10 days; or,
(3) Force reqd. to harvest 200 A. in 10 days = 15 men.

The *question* is stated as before:

Time reqd. for 5 men to harvest 80 A. = ( ) da.?

The following principles, which have been illustrated and explained, should be learned:

PRINCIPLES: *1. Every problem has at least two parts, a question (or requirement) and a basis.*

2. The parts of a problem may always be stated in the form of equations.

3. Every basis has as many forms as it has numbers related.

In each of the following examples, the *question* and *all forms of the basis* are stated in equations.

EXAMPLES.

1. James has \$720, and John has $\frac{3}{4}$ as much. How much has John?

Parts: (1) $\frac{3}{4}$ of James's money = \$( )? (Question.)
(2) $\frac{4}{4}$ of James's money = \$720; or,
\$720 = $\frac{4}{4}$ of James's money. (Basis.)

2. What will 10 hats cost, at \$2.40 each?

Parts: (1) Cost of 10 hats = \$( )? (Question.)
(2) Cost of 1 hat = \$2.40; or,
\$2.40 = cost of 1 hat. (Basis.)

3. 15% of a debt is \$630. Find the debt.

Parts: (1) 100% of debt = \$( )? (Question.)
(2) 15% of debt = \$630; or,
\$630 = 15% of debt. (Basis.)

NOTE.—The sign, %, is read *per cent* and means *hundredth* or *hundredths.* 10% = $\frac{10}{100}$, 25% = $\frac{25}{100}$, etc. *100% of anything is all of it.*

4. In $\frac{3}{11}$ of a load of corn, there are 6 bushels. How many bushels in 25 such loads?

Parts: (1) Amount of 25 loads = ( ) bu.? (Question.)
(2) Amount of $\frac{3}{11}$ of a load = 6 bu.; or,
6 bu. = amount of $\frac{3}{11}$ of a load. (Basis.)

5. $\frac{5}{8}$ of an article is worth $.75. What is $\frac{11}{12}$ of it worth?

Parts: (1) Value of $\frac{11}{12}$ of article = $( )? (Question.)
(2) Value of $\frac{5}{8}$ of article = $.75; or,
$.75 = value of $\frac{5}{8}$ of article. (Basis.)

6. If 6 men plow 45 acres in 3 days, how many acres will 1 man plow in 1 day?

Parts: (1) Amt. plowed by 1 man in 1 da. = ( ) A.? (Question.)
(2) Amt. plowed by 6 men in 3 da. = 45 A.; or,
Force reqd. to plow 45 A. in 3 da. = 6 men; or,
Time reqd. for 6 men to plow 45 A. = 3 da. (Basis.)

7. If 25 boxes of pens are worth $7.50, what is $\frac{1}{8}$ of a box worth?

Parts: (1) Value of $\frac{1}{8}$ box = $( )? (Question.)
(2) Value of 25 boxes = $7.50; or,
$7.50 = value of 25 boxes. (Basis.)

8. The volume of a solid 7 ft. long, 5 ft. wide and 4 ft. thick is 140 cubic feet. Find the volume of a solid 10 ft. long, 7 ft. wide and 5 ft. thick.

Parts: (1) Vol. of a solid 10 ft. l., 7 ft. w., 5 ft. th. = ( ) cu. ft.? (Question.)
(2) Vol. of a solid 7 ft. l., 5 ft. w., 4 ft. th. = 140 cu. ft.; or,
Length of a solid of 140 cu. ft., 5 ft. w., 4 ft. th. = 7 ft.; or,
Width of a solid of 140 cu. ft., 7 ft. l., 4 ft. th. = 5 ft.; or,
Thickness of a solid of 140 cu. ft., 7 ft. l., 5 ft. w. = 4 ft. (Basis.)

9. 21 is what part of 56?

Parts: (1) 21 = ( ) of 56? (Question.)
(2) 56 = all of 56; or,
All of 56 = 56. (Basis.)

NOTE.—"21 = ( ) of 56?" should be read, "21 equals *what part* of 56?"

10. 760 is how many times 38?

Parts: (1) 760 = ( ) × 38? (Question.)
(2) 38 = 1 × 38; or,
1 × 38 = 38. (Basis.)

EXERCISE XLIX.

State the question and all forms of the basis in each of the following:

1. Find 20% of $600.

2. What cost $\frac{3}{4}$ of a ton of coal at $3.80 per ton?

3. Clara is 12 years old; her mother is four times as old. How old is her mother?

4. Find 300% of $240.

5. $\frac{5}{8}$ of my salary is $375. What is my salary?

6. 25% of a certain debt is $1600. Find the debt.

7. If $\frac{11}{17}$ of a boat is worth $9900, what is the whole boat worth?

8. $8\frac{1}{3}$% of a certain sum is $50. What would 400% of that sum be?

9. If $\frac{2}{3}$ of a bolt of cloth is worth $3.20, what are 13 such bolts worth?

10. If $\frac{7}{8}$ of Mr. Brown's capital is $700, how much is $\frac{11}{16}$ of it?

11. I deposit 48% of my money, which is $960, in the bank, and invest 28% of my money in calves. How much do I spend for calves?

12. 360% of a number is 252. Find the number.

13. If 20 bu. of apples cost $18, find the price per bushel.

14. If 5 men can saw 20 cords of wood in 4 days, how many cords can 1 man saw in 1 day?

15. 120% of a number is 72. Find 35% of it.

16. If 11 men can set 660 pages of type in 6 days, how many pages will 21 men set in 8 days?

17. 240% of my age is 84 years. Find 380% of it.

18. 75 bu. of wheat are worth $60. What are 42 bu. worth?

19. A lawyer charges $35 for collecting $700. What % does he charge?

20. I bought an article for $5 and sold it so as to lose $3. What part of my investment did I lose?

21. I bought an article for \$5 and sold it for \$7.50. The selling price is what % of the cost price?

22. 5 times my money would be \$4620. How much have I?

23. 50 doz. eggs sell for \$4.50. Find the price per dozen.

24. Eggs are 9¢ per dozen. How many dozen can I buy for \$3.60?

25. Eggs are 9¢ per dozen. What will 30 dozen cost?

26. Reduce 64 pints to quarts. (**Basis omitted.**)

27. The area of a surface 1 ft. long, 1 ft. wide is 1 sq. ft. Find the length of a lot 150 ft. wide, containing 30000 sq. ft.

28. If 5 boys plow 12 acres in $1\frac{1}{2}$ days, in how many days will 12 boys plow 60 acres?

29. Find the volume of a stone 12 ft. long, 6 ft. wide, and 3 ft. thick. (**Basis omitted.**)

NOTE.—Preserve your work for use in Exercises L and LI.

72. Classification.—The problems of one basis, though varied in their forms of expression, when considered with reference to the relation of the *given part* to the *required part*, consist of no more than *ten different types;* and these may be classed into *four classes*, as follows:

CLASS 1. *Given a number, to find* (1) *a part of it, or* (2) *a multiple of it.*

EXAMPLES.

1. What is the value of $\frac{3}{5}$ of an article which cost \$7.20?

2. What will 50 books cost, at \$2.50 each?

CLASS 2. *Given a part of a number, to find* (1) *the number,* (2) *a multiple of it, or* (3) *a part of it.*

EXAMPLES.

1. 20% of a sale is \$540. What was the whole amount of the sale?

NOTE.—Any number of % less than 100% is a *part;* any number of % greater than 100% is a multiple.

2. $\frac{5}{13}$ of a certain farm is 25 acres. How many acres in 3 such farms?

3. $\frac{3}{4}$ of a certain article is worth \$.75. What is $\frac{4}{5}$ of it worth?

CLASS 3. *Given a multiple of a number, to find* (1) *the number,* (2) *a part of the number, or* (3) *another multiple of it.*

EXAMPLES.

1. If 35 hats sell for \$105, find the price of 1 hat.

2. At the rate of \$120 in 2 months, how much money can be earned in $\frac{3}{5}$ of a month?

3. 12 pounds troy equal 144 ounces; how many ounces do 7 pounds troy equal?

CLASS 4. *Given two numbers, to find* (1) *what part one is of the other, or* (2) *what multiple one is of the other.*

EXAMPLES.

1. A commission merchant charges \$96 for making a sale of \$3200. What per cent does he charge?

2. John has \$.50 and George has \$5. George's money is how many times John's?

It may be ascertained to which of the above 10 types a problem belongs by examining its parts. Thus, in example 2, Class 3,

Parts: (1) Earnings for $\frac{3}{5}$ mo. = \$( )? (Question.)
(2) Earnings for 2 mo. = \$120. (Basis.)

Here are given the earnings for 2 months, or 2 times a number; to find the earnings for $\frac{3}{5}$ month, or $\frac{3}{5}$ of the number. It gives a multiple of a number, to find a part of it.

EXERCISE L.

1. Determine to which of the 10 types each of the examples under Article 71 belongs.

2. Determine to which of the 10 types each problem of Exercise XLIX belongs.

C. PROBLEMS OF ONE BASIS—SOLUTION.

73. Two Methods of Solution.—Solving a Problem consists in commencing with a *given* (*or known*) *relation* and passing by means of an *authorized process* to the *required relation.* There are two methods employed in arithmetic for solving problems of one basis:

1. The **Equation Method,** in which *equations* are used throughout to express the steps of the solution. This method includes what by many authors is called "*Arithmetical Analysis.*"

2. The **Proportion Method,** in which the solution is obtained by forming and solving a *proportion.*

1. THE EQUATION METHOD.

74. Arranging the Parts of a Problem.—For the sake of convenience and system in solution, the following rules should be observed in arranging the parts of a problem:

RULES: *1. The question, or requirement, of a problem should be stated in the form of an equation with only the blank term on the right of the sign of equality.*

2. The basis of the problem should be stated in that form which agrees in arrangement with the question, or requirement, as stated according to Rule 1.

NOTE.—It is very necessary that the pupil should learn to state by the above rules the parts of a problem without hesitation.

In the following examples, the parts of the problems are properly arranged for solution.

EXAMPLES.

1. If 8 books cost $4, find the cost of 3 books.

Parts: (1) Cost of 3 books = $( )? (Question.)
(2) Cost of 8 books = $4. (Basis.)

2. If 8 books cost \$4, how many books can be bought for \$10?

Parts: (1) \$10 = cost of ( ) books? (Question.)
(2) \$4 = cost of 8 books. (Basis.)

NOTE.—Compare the arrangement of parts in Nos. 1 and 2.

3. Reduce 9 feet to inches. (1 foot = 12 inches.)

Parts: (1) 9 ft. = ( ) in.? (Question.)
(2) 1 ft. = 12 in. (Basis.)

4. Reduce 72 inches to feet. (1 inch = $\frac{1}{12}$ foot.)

Parts: (1) 72 in. = ( ) ft.? (Question.)
(2) 1 in. = $\frac{1}{12}$ ft. (Basis.)

NOTE.—Compare the arrangement of parts in Nos. 3 and 4.

5. I owe a debt of \$720, and can pay 45% of it. How much can I pay?

Parts: (1) 45% of debt = \$( )? (Question.)
(2) 100% of debt = \$720. (Basis.)

NOTE.—Remember that 100% of anything is all of it.

6. I owe \$120 and pay \$40. What % of the debt do I pay?

Parts: (1) \$40 = ( )% of debt? (Question.)
(2) \$120 = 100% of debt. (Basis.)

NOTE.—Compare the arrangement of parts in Nos. 5 and 6.

7. Find the interest on \$500 for 2 years at 8%. (Basis omitted.)

Parts: (1) Int. on \$500 at 8% for 2 yr. = \$( )? (Question.)
(2) Int. on \$1 at 1% for 1 yr. = \$.01. (Basis.)

8. Find the principal required to produce \$80 interest in 2 years at 8%. (Basis omitted.)

Parts: (1) Prin. reqd. to produce \$80 in 2 yr. at 8% = \$( )? (Question.)
(2) Prin. reqd. to produce \$.01 in 1 yr. at 1% = \$1. (Basis.)

9. Find the rate required for \$500 to produce \$80 in 2 years. (Basis omitted.)

Parts: (1) Rate reqd. for \$500 to produce \$80 in 2 yr. = ( )%? (Question.)
(2) Rate reqd. for \$1 to produce \$.01 in 1 yr. = 1%. (Basis.)

10. Find the time required for $500 to produce $80 at 8%. (Basis omitted.)

Parts: (1) Time reqd. for $500 to produce $80 at 8% = ( ) yr.? (Question.)
(2) Time reqd. for $1 to produce $.01 at 1% = 1 yr. (Basis.)

NOTE.— Compare the arrangement of parts in Nos. 7 to 10.

11. Find the area of a floor 12 feet long and 8 feet wide. (Basis omitted.)

Parts: (1) Area of a surface 12 ft. l., 8 ft. w. = ( ) sq. ft.? (Question.)
(2) Area of a surface 1 ft. l., 1 ft. w. = 1 sq. ft. (Basis.)

12. Find the length of a floor 12 feet wide, containing 240 square feet. (Basis omitted.)

Parts: (1) Length of a surface of 240 sq. ft., 12 ft. w. = ( ) ft.? (Question.)
(2) Length of a surface of 1 sq. ft., 1 ft. w. = 1 ft. (Basis.)

13. Find the width of a surface of 108 square feet, 12 feet long. (Basis omitted.)

Parts: (1) Width of a surface of 108 sq. ft., 12 ft. l. = ( ) ft ? (Question.)
(2) Width of a surface of 1 sq. ft., 1 ft. l. = 1 ft. (Basis.)

NOTE.—Compare the arrangement of parts in Nos. 11 to 13.

It should be remembered that, in stating a problem, the basis is omitted only when it is supposed that the pupil already has the knowledge necessary to enable him to state the basis for himself.

EXERCISE LI.

I. *Arrange for solution the parts of each problem in Exercise XLIX.*

II. *Arrange for solution the parts of each of the following:*

1. 150 ounces is $\frac{3}{4}$ of what number ?

2. Reduce 120 feet to inches.

3. Reduce 228 inches to feet.

4. If 5 books cost $25, find the cost of 12 books.

5. Basis same as in No. 4. Find how many books can be bought for $60.

6. If a boy earns $.50 in 5 hours, what will he earn in 13 hours?

7. Find how long it will take $620 at 6% to gain $9 interest.

8. Find how much interest $740 will gain at 8% in $3\frac{1}{3}$ years.

9. Basis same as in No. 6. How long can I hire him for $3.50?

10. Find the area of a field 20 rods long and 12 rods wide.

11. A field of 240 sq. rd. is 8 rods wide. How long is it?

12. If 5 men excavate 20 tons of dirt in 2 days, how many tons will 12 men excavate in 24 days?

13. Basis same as in No. 12. In what time will 8 men excavate 120 tons?

14. 25% of my money is $250. How much money have I?

15. Basis same as in No. 14. What per cent of my money is $450?

16. Find the volume of a rectangular cistern 8 ft. long, 5 ft. wide, and 7 ft. deep.

17. At what rate will $800 produce $20 interest in 5 months?

18. What is the time, when $\frac{2}{3}$ of the time past noon is 4 hours?

19. The volume of a rectangular solid is 360 cu. ft., the length 15 ft., the width 8 ft. Find the thickness.

20. A can do a piece of work in 20 days. How much of it can he do in 9 days?

21. Basis same as in No. 20. How long will A be in doing $\frac{11}{40}$ of the work?

22. I bought an article for $5, and sold it for $\frac{5}{4}$ of the cost. Find the selling price.

23. 42 is $\frac{3}{4}$ of what number?

24. 400 is how many times 25?

75. The Process of Solving Problems.—After the parts of a problem have been arranged according to the rules given in the last article, the process of solving by equations consists of one or both of the following steps:

STEPS: *1. Perform such operation, or operations, upon the basis as will change it to an equation, the numerical value of whose first member is unity.*

2. Perform such operation, or operations, upon the equation, obtained in Step 1, as will change it to an equation whose first member is identical with the first member of the question.

If the numerical value of the first member of the *question* is unity, the *first step* only is used; if the numerical value of the first member of the *basis* is unity, the *second step* only is used.

Example: If 24 hats cost \$12, how many hats can be bought for \$16?

Solution: (1) \$16 = cost of ( ) hats? (Question.)
(2) \$12 = cost of 24 hats. (Basis.)
$\frac{1}{12}$ of (2) = (3) \$1 = cost of 2 hats.
16 × (3) = (4) \$16 = cost of 32 hats, answer.

76. Integral Solutions.—A solution whose equations contain only integers in their first members is an *integral solution.*

NOTE.—Integral solutions are not always free from fractions (see Example 4, page 117); but fractions may occur in the second members of the equations only.

EXAMPLES.

1. Find the cost of 10 mats at \$3 each.

Solution: (1) Cost of 10 mats = \$( )? (Question.)
(2) Cost of 1 mat = \$3. (Basis.)
10 × (2) = (3) Cost of 10 mats = \$30, answer.

2. If 10 mats cost \$30, what will 1 mat cost ?

Solution: (1) Cost of 1 mat = \$( ) ? (Question.)
(2) Cost of 10 mats = \$30. (Basis.)
$\frac{1}{10}$ of (2) = (3) Cost of 1 mat = \$3, answer.

3. If 10 mats cost \$30, what cost 17 mats ?

Solution: (1) Cost of 17 mats = \$( ) ? (Question.)
(2) Cost of 10 mats = \$30. (Basis.)
$\frac{1}{10}$ of (2) = (3) Cost of 1 mat = \$3.
17 × (3) = (4) Cost of 17 mats = \$51, answer.

4. If 10 mats cost \$30, how many can be bought for \$63 ?

Solution: (1) \$63 = cost of ( ) mats ? (Question.)
(2) \$30 = cost of 10 mats. (Basis.)
$\frac{1}{30}$ of (2) = (3) \$1 = cost of $\frac{1}{3}$ mat.
63 × (3) = (4) \$63 = cost of $\frac{63}{3}$ or 21 mats, answer.

5. A man has \$75, and loans \$35. What part of his money does he loan ?

Solution: (1) \$35 = ( ) of his money ? (Question.)
(2) \$75 = all of his money. (Basis.)
$\frac{1}{75}$ of (2) = (3) \$1 = $\frac{1}{75}$ of his money.
35 × (3) = (4) \$35 = $\frac{35}{75}$, or $\frac{7}{15}$ of his money, answer.

6. 12 is what part of 40 ?

Solution: (1) 12 = ( ) of 40 ? (Question.)
(2) 40 = all of 40. (Basis.)
$\frac{1}{40}$ of (2) = (3) 1 = $\frac{1}{40}$ of 40.
12 × (3) = (4) 12 = $\frac{12}{40}$, or $\frac{3}{10}$ of 40, answer.

7. 40 is how many times 12 ?

Solution: (1) 40 = ( ) × 12 ? (Question.)
(2) 12 = 1 × 12. (Basis.)
$\frac{1}{12}$ of (2) = (3) 1 = $\frac{1}{12}$ of 12.
40 × (3) = (4) 40 = $\frac{40}{12}$, or $3\frac{1}{3} \times 12$, answer.

8. If 5 men can do a piece of work in 42 days, in what time can 14 men do it ?

Solution: (1) Time reqd. by 14 men = ( ) da.? (Question.)
(2) Time reqd. by 5 men = 42 da. (Basis.)
$5 \times (2) = (3)$ Time reqd. by 1 man = 210 da.
$\frac{1}{14}$ of $(3) = (4)$ Time reqd. by 14 men = 15 da., answer.

9. Find the area of a surface 16 rods long and 10 rods wide.

Solution: (1) Area of a surf. 16 rd. l., 10 rd. w. = ( ) sq. rd.? (Question.)
(2) Area of a surf. 1 rd. l., 1 rd. w. = 1 sq. rd. (Basis.)
$16 \times (2) = (3)$ Area of a surf. 16 rd. l., 1 rd. w. = 16 sq. rd.
$10 \times (3) = (4)$ Area of a surf. 16 rd. l., 10 rd. w. = 160 sq. rd., answer.

10. Find the length of a surface of 160 square rods, 10 rods wide.

Solution: (1) Length of a surf. of 160 sq. rd., 10 rd. w. = ( ) rd.? (Question.)
(2) Length of a surf. of 1 sq. rd., 1 rd. w. = 1 rd. (Basis.)
$160 \times (2) = (3)$ Length of a surf. of 160 sq. rd., 1 rd. w. = 160 rd.
$\frac{1}{10}$ of $(3) = (4)$ Length of a surf. of 160 sq. rd., 10 rd. w. = 16 rd., answer.

11. Find the width of a surface of 160 square rods, 16 rods long.

Solution: (1) Width of a surf. of 160 sq. rd., 16 rd. l. = ( ) rd.? (Question.)
(2) Width of a surf. of 1 sq. rd., 1 rd. l. = 1 rd. (Basis.)
$160 \times (2) = (3)$ Width of a surf. of 160 sq. rd., 1 rd. l. = 160 rd.
$\frac{1}{16}$ of $(3) = (4)$ Width of a surf. of 160 sq. rd., 16 rd. l. = 10 rd., answer.

12. If 5 men build 100 rods of fence in 10 days, how many rods can 3 men build in 7 days?

Solution: (1) Amt. built by 3 men in 7 da. = ( ) rd.? (Question.)
(2) Amt. built by 5 men in 10 da. = 100 rd. (Basis.)
$\frac{1}{5}$ of $(2) = (3)$ Amt. built by 1 man in 10 da. = 20 rd.
$\frac{1}{10}$ of $(3) = (4)$ Amt. built by 1 man in 1 da. = 2 rd.
$3 \times (4) = (5)$ Amt. built by 3 men in 1 da. = 6 rd.
$7 \times (5) = (6)$ Amt. built by 3 men in 7 da. = 42 rd., answer.

13. If 5 men build 100 rods of fence in 10 days, how many men can build 42 rods in 7 days?

Solution: (1) Force reqd. to build 42 rd. in 7 da. = ( ) men? (Question.)
(2) Force reqd. to build 100 rd. in 10 da. = 5 men. (Basis.)
$\frac{1}{100}$ of $(2) = (3)$ Force reqd. to build 1 rd. in 10 da. = $\frac{5}{100}$, or $\frac{1}{20}$ man.

10×(3)=(4) Force reqd. to build 1 rd. in 1 da. = $\frac{10}{20}$, or $\frac{1}{2}$ man.
42×(4)=(5) Force reqd. to build 42 rd. in 1 da. = 21 men.
$\frac{1}{7}$ of (5)=(6) Force reqd. to build 42 rd. in 7 da. = 3 men, answer.

14. If 5 men can build 100 rods of fence in 10 days, how many days will be required for 3 men to build 42 rods?

Solution: (1) Time reqd. for 3 men to build 42 rd. = ( ) da.? (Question.)
(2) Time reqd. for 5 men to build 100 rd. = 10 da. (Basis.)
5×(2)=(3) Time reqd. for 1 man to build 100 rd. = 50 da.
$\frac{1}{100}$ of (3)=(4) Time reqd. for 1 man to build 1 rd. = $\frac{1}{2}$ da.
$\frac{1}{3}$ of (4)=(5) Time reqd. for 3 men to build 1 rd. = $\frac{1}{6}$ da.
42×(5)=(6) Time reqd. for 3 men to build 42 rd. = 7 days, answer.

15. Reduce 42 gallons to quarts.

Solution: (1) 42 gal. = ( ) qt.? (Question.)
(2) 1 gal. = 4 qt. (Basis.)
42×(2)=(3) 42 gal. = 168 qt., answer.

16. Reduce 172 quarts to gallons.

Solution: (1) 172 qt. = ( ) gal.? (Question.)
(2) 1 qt. = $\frac{1}{4}$ gal. (Basis.)
172×(2)=(3) 172 qt. = 43 gal., answer.

17. Find the interest on $500 at 6% per annum for 3 years.

Solution: (1) Int. on $500 at 6% for 3 yr. = $( )? (Question.)
(2) Int. on $1 at 1% for 1 yr. = $.01. (Basis.)
500×(2)=(3) Int. on $500 at 1% for 1 yr. = $5.
6×(3)=(4) Int. on $500 at 6% for 1 yr. = $30.
3×(4)=(5) Int. on $500 at 6% for 3 yr. = $90, answer.

18. Find the rate required for $500 to gain $90 in 3 years.

Solution: (1) Rate reqd. for $500 to gain $90 in 3 yr. = ( )%? (Question.)
(2) Rate reqd. for $1 to gain $.01 in 1 yr. = 1%. (Basis.)
$\frac{1}{500}$ of (2)=(3) Rate reqd. for $500 to gain $.01 in 1 yr. = $\frac{1}{500}$%.
9000×(3)=(4) Rate reqd. for $500 to gain $90 in 1 yr. = 18%.
$\frac{1}{3}$ of (4)=(5) Rate reqd. for $500 to gain $90 in 3 yr. = 6%, answer.

19. Find the time required for $500 to gain $90 at 6% per annum.

Solution: (1) Time reqd. for \$500 to gain \$90 at 6% = ( ) yr.? (Question.)
(2) Time reqd. for \$1 to gain \$.01 at 1% = 1 yr. (Basis.)
$\frac{1}{500}$ of (2) = (3) Time reqd. for \$500 to gain \$.01 at 1% = $\frac{1}{500}$ yr.
9000 × (3) = (4) Time reqd. for \$500 to gain \$90 at 1% = 18 yr.
$\frac{1}{6}$ of (4) = (5) Time reqd. for \$500 to gain \$90 at 6% = 3 yr., answer.

20. What principal will gain \$90 interest in 3 years at 6% per annum?

Solution: (1) Prin. reqd. to gain \$90 in 3 yr. at 6% = \$( )? (Question.)
(2) Prin. reqd. to gain \$.01 in 1 yr. at 1% = \$1. (Basis.)
9000 × (2) = (3) Prin. reqd. to gain \$90 in 1 yr. at 1% = \$9000.
$\frac{1}{3}$ of (3) = (4) Prin. reqd. to gain \$90 in 3 yr. at 1% = \$3000.
$\frac{1}{6}$ of (4) = (5) Prin. reqd. to gain \$90 in 3 yr. at 6% = \$500, answer.

EXERCISE LII.

1. What will 20 hats cost at \$2.50 each?

2. 21 is what part of 49?

3. What cost 22 yards of muslin at 10¢ per yard? 18 yards of flannel at 40¢ per yard?

4. What are 40 acres of land worth at \$50.20 per acre?

5. The light from the sun reaches the earth in 8 minutes, and it travels at the rate of 188000 miles per second. Find the distance to the sun.

6. A has \$840; B has 7 times as much; C has 5 times as much as B. How much has B, and C?

7. What is the area of a surface 120 feet long and 30 feet wide?

8. A can travel on a bicycle 4 times as fast as he can walk. If he can walk 2 miles an hour, how fast can he go on a bicycle?

9. 4 acres is what part of 10 acres?

10. I charge \$75 for collecting \$750: what part of the amount collected will pay me for collecting?

11. If 1 man can build a certain fence in 12 days, how long will it take 4 men to build it?

12. An area of 140 sq. ft. is 7 ft. wide. How long is it?

13. An area of 150 sq. ft. is 50 ft. long. How wide is it?

14. A's money, $500, is how many times B's money, $125?

15. Find the volume of a solid 9 ft. long, 7 ft. wide, and 10 ft. high.

16. A volume of 630 cu. ft. is 9 ft. long and 7 ft. wide. How high is it?

17. A volume of 630 cu. ft. is 10 ft. high and 7 ft. wide. How long is it?

18. $6800 is how many times $260?

19. The distance to the sun is 90240000 miles. Light travels from the sun to the earth in 8 minutes: what is the velocity of light per second?

20. A lady earns $45 per month (20 days). How much is that per day?

21. If 20 men harvest 180 acres in 5 days, how much does each man harvest per day?

22. Three leaps of a fox is as far as two leaps of a hound. Reduce 60 fox-leaps to hound-leaps.

Question: 60 f. l. = ( ) h. l.?

23. Basis same as in No. 22. Reduce 60 hound-leaps to fox-leaps.

24. The minute-hand of a clock travels 12 times as fast as the hour-hand. How long will the hour-hand be in traveling 20 minute-spaces?

Question: Time reqd. by h. h. to travel 20 min. spaces = ( ) min.?

25. How long will the minute-hand be in traveling 3 hour-spaces?

26. The sun passes over 15 degrees of longitude in 1 hour: how many hours will it be in passing over 100 degrees?

27. If $4.86 buys an exchange on London for £1, how large an exchange can I buy for $9720?

28. Find the interest on $720 for 5 years at 8% per annum.

29. What principal will produce $168 interest in 4 years at 7% per annum?

30. At what rate will $540 bear $81 in 3 years?

77. Fractional Solutions.—Fractional solutions are those in which the first members of all equations are *fractions* or are expressed as *fractions.*

PREPARATORY STEP: *Reduce the fractions in the first members of the parts of the problem to a common denominator.*

EXAMPLES.

1. $\frac{3}{4}$ of A's money is $4500. Find $\frac{4}{5}$ of it.

Solution: (1) Reduced to L. C. D., $\frac{3}{4}$ and $\frac{4}{5} = \frac{15}{20}$ and $\frac{16}{20}$.
(2) $\frac{16}{20}$ of A's money = $( )? (Question.)
(3) $\frac{15}{20}$ of A's money = $4500. (Basis.)
$\frac{1}{15}$ of (3) = (4) $\frac{1}{20}$ of A's money = $300.
16 × (4) = (5) $\frac{16}{20}$ of A's money = $4800, answer.

2. If 1 ton of hay costs $5, what costs $\frac{7}{10}$ of a ton?

Solution: (1) Cost of $\frac{7}{10}$ T. = $( )? (Question.)
(2) Cost of $\frac{10}{10}$ T. = $5. (Basis.)
$\frac{1}{10}$ of (2) = (3) Cost of $\frac{1}{10}$ T. = $.50.
7 × (3) = (4) Cost of $\frac{7}{10}$ T. = $3.50, answer.

3. Find $\frac{3}{8}$ of $4000.

Solution: (1) $\frac{3}{8}$ of $4000 = $( )? (Question.)
(2) $\frac{8}{8}$ of $4000 = $4000. (Basis.)
$\frac{1}{8}$ of (2) = (3) $\frac{1}{8}$ of $4000 = $500.
3 × (3) = (4) $\frac{3}{8}$ of $4000 = $1500, answer.

4. 342 is $\frac{3}{4}$ of what number?

Solution: (1) $\frac{4}{4}$ of No. = ( )? (Question.)
(2) $\frac{3}{4}$ of No. = 342. (Basis.)
$\frac{1}{3}$ of (2) = (3) $\frac{1}{4}$ of No. = 114.
4 × (3) = (4) $\frac{4}{4}$ of No. = 456, answer.

5. What number added to $\frac{5}{9}$ of itself will give 560?

Solution: (1) $\frac{9}{9}$ of No. = ( )? (Question.)
(2) $\frac{9}{9}$ of No. + $\frac{5}{9}$ of No. = 560. (Basis.)
(2) = (3) $\frac{14}{9}$ of No. = 560.
$\frac{1}{14}$ of (3) = (4) $\frac{1}{9}$ of No. = 40.
$9 \times (4) = (5)$ $\frac{9}{9}$ of No. = 360, answer.

6. What number less its $\frac{4}{7}$ leaves 510?

Solution: (1) $\frac{7}{7}$ of No. = ( )? (Question.)
(2) $\frac{7}{7}$ of No. − $\frac{4}{7}$ of No. = 510. (Basis.)
(2) = (3) $\frac{3}{7}$ of No. = 510.
$\frac{1}{3}$ of (3) = (4) $\frac{1}{7}$ of No. = 170.
$7 \times (4) = (5)$ $\frac{7}{7}$ of No. = 1190, answer.

7. Find that number whose $\frac{5}{8}$ added to its $\frac{2}{3}$ will give 1240.

Solution: (1) Reduced to L. C. D., $\frac{5}{8}$ and $\frac{2}{3}$ = $\frac{15}{24}$ and $\frac{16}{24}$.
(2) $\frac{24}{24}$ of No. = ( )? (Question.)
(3) $\frac{15}{24}$ of No. + $\frac{16}{24}$ of No. = 1240. (Basis.)
(3) = (4) $\frac{31}{24}$ of No. = 1240.
$\frac{1}{31}$ of (4) = (5) $\frac{1}{24}$ of No. = 40.
$24 \times (5) = (6)$ $\frac{24}{24}$ of No. = 960, answer.

8. Find that number whose $\frac{7}{11}$ less its $\frac{1}{4}$ leaves 850.

Solution: (1) Reduced to L. C. D., $\frac{7}{11}$ and $\frac{1}{4}$ = $\frac{28}{44}$ and $\frac{11}{44}$.
(2) $\frac{44}{44}$ of No. = ( )? (Question.)
(3) $\frac{28}{44}$ of No. − $\frac{11}{44}$ of No. = 850. (Basis.)
(3) = (4) $\frac{17}{44}$ of No. = 850.
$\frac{1}{17}$ of (4) = (5) $\frac{1}{44}$ of No. = 50.
$44 \times (5) = (6)$ $\frac{44}{44}$ of No. = 2200, answer.

9. 720 is $\frac{2}{3}$ more than what number?

Solution: (1) $\frac{3}{3}$ of No. = ( )? (Question.)
(2) $\frac{3}{3}$ of No. + $\frac{2}{3}$ of No. = 720. (Basis.)
(2) = (3) $\frac{5}{3}$ of No. = 720.
$\frac{1}{5}$ of (3) = (4) $\frac{1}{3}$ of No. = 144.
$3 \times (4) = (5)$ $\frac{3}{3}$ of No. = 432, answer.

10. 102 is $\frac{2}{5}$ less than what number?

Solution: (1) $\frac{5}{5}$ of No. = ( )? (Question.)
(2) $\frac{5}{5}$ of No. − $\frac{2}{5}$ of No. = 102. (Basis.)
(2) = (3) $\frac{3}{5}$ of No. = 102.
$\frac{1}{3}$ of (3) = (4) $\frac{1}{5}$ of No. = 34.
$5 \times (4) = (5)$ $\frac{5}{5}$ of No. = 170, answer.

11. $\frac{3}{4}$ is what part of $\frac{7}{8}$?

Solution: (1) Reduced to L. C. D., $\frac{3}{4}$ and $\frac{7}{8} = \frac{6}{8}$ and $\frac{7}{8}$.

(2) $\frac{6}{8} = (\ \ )$ of $\frac{7}{8}$? (Question.)

(3) $\frac{7}{8} =$ all of $\frac{7}{8}$. (Basis.)

$\frac{1}{7}$ of (3) = (4) $\frac{1}{8} = \frac{1}{7}$ of $\frac{7}{8}$.

$6 \times$ (4) = (5) $\frac{6}{8} = \frac{6}{7}$ of $\frac{7}{8}$.

$\therefore$ $\frac{3}{4}$ is $\frac{6}{7}$ of $\frac{7}{8}$.

12. .12 is how many times .08?

Solution: (1) $.12 = (\ \ ) \times .08$? (Question.)

(2) $.08 = 1 \times .08$. (Basis.)

$\frac{1}{8}$ of (2) = (3) $.01 = \frac{1}{8}$ of .08.

$12 \times$ (3) = (4) $.12 = \frac{12}{8}$, or $1\frac{1}{2} \times .08$, answer.

EXERCISE LIII.

1. $\frac{4}{5}$ of a number is 1240. Find $\frac{7}{8}$ of the number.

2. My salary is \$765. If I spend $\frac{5}{9}$ of it, what amount do I spend?

3. A man can build $\frac{4}{5}$ of a wall in 12 days. In what time can he build $\frac{2}{3}$ of it?

4. In selling, a merchant lost $\frac{4}{11}$ of the cost on an article which cost him \$20.24. How much did he lose? What did the article sell for?

5. A has $\frac{5}{11}$ as much money as B. If they both have \$6400, how much has B? How much has A?

Basis: $\frac{5}{11}$ of B's money + $\frac{11}{11}$ of B's money = \$6400. (Why?)

6. 48 is $\frac{3}{13}$ of what number?

7. $\frac{3}{19}$ of a number is 1542. Find the number.

8. $\frac{5}{12}$ of a bolt of muslin costs \$5. What will the whole bolt cost?

9. I charge $\frac{1}{20}$ of a debt for collecting it. If I receive \$240, what was the whole debt?

10. \$950 is $\frac{7}{12}$ more than what number?

11. \$475 is $\frac{7}{12}$ less than what number?

12. Find the number whose $\frac{5}{9}$ added to its $\frac{1}{15}$ makes \$920.

13. Find the number whose $\frac{11}{14}$ less its $\frac{2}{21}$ leaves \$2030.

14. What number added to its $\frac{5}{12}$ gives 136?

15. What number less its $\frac{4}{5}$ leaves 75?

16. What is $\frac{3}{8}$ of a ton of hay worth at \$3.20 per ton?

17. If $\frac{5}{12}$ of a farm is worth \$1750, what is the whole farm worth?

18. B can do $\frac{3}{4}$ of a piece of work in $\frac{3}{7}$ of a day. What part of it can he do in $\frac{8}{9}$ of a day?

19. $1\frac{1}{2}$ times a number exceeds its $\frac{3}{11}$ by 54. Find $3\frac{1}{2}$ times the number.

20. A loaned B $\frac{5}{9}$ of his money and had \$480 left. How much money had A before making the loan?

21. The sum of two numbers is 54; the smaller is $\frac{1}{2}$ of the larger. Find the numbers.

Basis: $\frac{2}{2}$ of larger $+ \frac{1}{2}$ of larger $= 54$. (Why?)

78. Reciprocals and Their Use.—The **Reciprocal** of a number is one divided by that number.

EXAMPLES.

1. The reciprocal of $15 = \frac{1}{15}$.

NOTE.—The reciprocal of an integer is expressed as a common fraction whose numerator is 1 and whose denominator is the given integer.

2. The reciprocal of $.03 = \frac{1}{.03}$.

3. The reciprocal of $3.25 = \frac{1}{3.25}$.

NOTE.—The reciprocal of a decimal is expressed as a common fraction whose numerator is 1 and whose denominator is the given decimal.

4. The reciprocal of $\frac{5}{7} = 1 \div \frac{5}{7} = 1 \times \frac{7}{5} = \frac{7}{5}$.

NOTE.—The reciprocal of a common fraction is expressed by inverting the fraction.

5. The reciprocal of $3\frac{1}{3}$, or $\frac{10}{3} = \frac{3}{10}$.

NOTE.—Reduce to an improper fraction and proceed as in No. 4 above.

After stating the parts of a problem for solution often the first step is to proceed from what is given, to unity (Step 1, p. 116). This can always be done directly, either in integral or fractional problems, by employing the following principle:

PRINCIPLE: *The product of a number by its reciprocal equals 1.*

EXAMPLES.

1. If 20 books cost \$50, what cost 1 book?

Solution: (1) Cost of 1 book = \$( )? (Question.)
(2) Cost of 20 books = \$50. (Basis.)
$\frac{1}{20} \times (2) = (3)$ Cost of 1 book = \$2.50, answer.

NOTE.—Did we multiply by the reciprocal of 20? We have heretofore used "of" instead of "×"; is "of" correct? (See Article 42.)

2 If $3\frac{1}{3}$ cords of wood are worth \$8, what is 1 cord worth?

Solution: (1) Price of 1 cord = \$( )? (Question.)
(2) Price of $3\frac{1}{3}$ cords = \$8. (Basis.)
$\frac{3}{10}$ of (2) = (3) Price of 1 cord = \$2.40, answer.

NOTE.—$\frac{3}{10}$ is the reciprocal of $3\frac{1}{3}$.

3. At \$1.25 per bu., how much wheat can be bought for \$22.50?

Solution: (1) \$22.50 = cost of ( ) bu.? (Question.)
(2) \$1.25 = cost of 1 bu. (Basis.)
$\frac{1}{1.25} \times (2) = (3)$ \$1 = cost of $\frac{1}{1.25}$ bu.
$22.50 \times (3) = (4)$ \$22.50 = cost of $\frac{22.50}{1.25}$, or 18 bu., answer.

NOTE.— $\frac{1}{1.25}$ is the reciprocal of 1.25.

4. $12\frac{1}{2}$ is what part of $37\frac{1}{2}$?

Solution: (1) $12\frac{1}{2}$ = ( ) of $37\frac{1}{2}$? (Question.)
(2) $37\frac{1}{2}$ = all of $37\frac{1}{2}$. (Basis.)
$\frac{2}{75}$ of (2) = (3) $1 = \frac{2}{75}$ of $37\frac{1}{2}$.
$12\frac{1}{2} \times (3) = (4)$ $12\frac{1}{2} = \frac{\cancel{25}}{\cancel{2}} \times \frac{\cancel{2}}{\underset{3}{\cancel{75}}}$, or $\frac{1}{3}$ of $37\frac{1}{2}$, answer.

NOTE.—$\frac{2}{75}$ is the reciprocal of $37\frac{1}{2}$.

EXERCISE LIV.

Solve the problems in Exercise LIII by using reciprocals instead of the fractional solution there given. Thus,

1. *Solution:* (1) $\frac{7}{8}$ of No. = ( )? (Question.)
(2) $\frac{4}{5}$ of No. = 1240. (Basis.)
$\frac{5}{4}$ of (2) = (3) $1 \times$ No. = 1550.
$\frac{7}{8}$ of (3) = (4) $\frac{7}{8}$ of No. = $1356\frac{1}{4}$, answer.

79. Solutions Shortened.—While it is strenuously urged that the pupil should always follow out a *complete* solution for every problem, it will become monotonous and even repulsive to the pupil to be compelled always to go the "long way around," when he can grasp completely a more direct solution.

Since solving a problem consists in commencing with a *known condition* and passing by means of a *known authorized process* from that *known condition* to the *unknown* (*required*) *condition*, then it follows that both the *known condition* and the *process employed* may depend largely upon the knowledge and ability of the person taught. The following suggestions will aid in shortening many solutions:

(1) Common fractions are usually to be preferred to decimals. Learn the following table:

EQUIVALENTS.

$\$.06\frac{1}{4} = \$\frac{1}{16}$
$\$.08\frac{1}{3} = \$\frac{1}{12}$
$\$.10 = \$\frac{1}{10}$
$\$.12\frac{1}{2} = \$\frac{1}{8}$
$\$.16\frac{2}{3} = \$\frac{1}{6}$
$\$.20 = \$\frac{1}{5}$
$\$.25 = \$\frac{1}{4}$
$\$.33\frac{1}{3} = \$\frac{1}{3}$
$\$.50 = \$\frac{1}{2}$

EXAMPLES.

1. At $16\frac{2}{3}$¢ each, how many knives can be bought for $3.50?

Solution: (1) $\$3\frac{1}{2}$ = cost of ( ) knives? (Question.)
(2) $\$\frac{1}{6}$ = cost of 1 knife. (Basis.)
$6 \times (2) = (3)$ \$1 = cost of 6 knives.
$3\frac{1}{2} \times (3) = (4)$ $\$3\frac{1}{2}$ = cost of 21 knives, answer.

2. What will 220 dozen eggs sell for, at $12\frac{1}{2}$¢ per dozen?

Solution: (1) Price of 220 doz. = \$( )? (Question.)
(2) Price of 1 doz. = $\$\frac{1}{8}$. (Basis.)
$220 \times (2) = (3)$ Price of 220 doz. = $\$27\frac{1}{2}$, or \$27.50, answer.

(2) If the first member of the basis is a known factor of the first member of the question, one step only is needed in solving the problem.

EXAMPLES.

1. 4 times A's money is \$800. If B has 12 times as much money as A, how much has B?

Solution: (1) $12 \times$ A's money = \$( )? (Question.)
(2) $4 \times$ A's money = \$800. (Basis.)
$3 \times (2) = (3)$ $12 \times$ A's money = \$2400, answer.

2. Reduce 128 ounces avoirdupois to pounds.

Solution: (1) 128 oz. = ( ) lb.? (Question.)
(2) 16 oz. = 1 lb. (Basis.)
$8 \times (2) = (3)$ 128 oz. = 8 lb., answer.

NOTE.—Here the basis of solution is not given; and may be stated in either of the forms: 16 oz. = 1 lb., or 1 oz. = $\frac{1}{16}$ lb. As the number to be reduced (128 oz.) is a multiple of 16 oz., the first form is as good as the second, if not better. But if the number to be reduced had not been a multiple of 16 oz., the second form would have been the better, since it starts with unity in its first member.

(3) If the first member of the basis is a known multiple of the first member of the question, one step only is needed in solving the problem.

EXAMPLES.

1. If \$85 will buy 30 books, how many books can be bought for \$17?

Solution: (1) \$17 = price of ( ) books? (Question.)
(2) \$85 = price of 30 books. (Basis.)
$\frac{1}{5}$ of (2) = (3) \$17 = price of 6 books, answer.

2. If 5 men can do a piece of work in 45 days, in what time can 15 men do the same work?

Solution: (1) Time reqd. by 15 men = ( ) da.? (Question.)
(2) Time reqd. by 5 men = 45 da. (Basis.)
$\frac{1}{3}$ of (2) = (3) Time reqd. by 15 men = 15 da., answer.

3. A bought two tracts of land at a uniform price per acre. The first was 80 rods long and 40 rods wide, and cost \$4000. What did the other cost, which was 20 rods long and 8 rods wide?

Solution: (1) Cost of a tract 20 rd. l., 8 rd. w. = \$( )? (Question.)
(2) Cost of a tract 80 rd. l., 40 rd. w. = \$4000. (Basis.)
$\frac{1}{4}$ of (2) = (3) Cost of a tract 20 rd. l., 40 rd. w. = \$1000.
$\frac{1}{5}$ of (3) = (4) Cost of a tract 20 rd. l., 8 rd. w. = \$200, answer.

(4) When the basis of a problem is omitted, the pupil should be permitted to use whatever basis he can readily understand.

EXAMPLE.

Find the interest on \$500 for $2\frac{1}{2}$ years at 6% per annum.

Solution: (1) Int. on \$500 at 6% for $2\frac{1}{2}$ yr. = \$( )? (Question.)
(2) Int. on \$500 at 6% for 1 yr. = \$30. (Basis.)
$2\frac{1}{2}\times$(2) = (3) Int. on \$500 at 6% for $2\frac{1}{2}$ yr. = \$75, answer.

NOTE.—Most advanced pupils will readily see that the "*Interest on \$500 for 1 yr. at 6% = \$30*"; others may be able only to see that "*Interest on \$500 for 1 yr. at 1% = \$5*"; while the beginner will have to commence with "*Interest on \$1 for 1 yr. at 1% = \$.01.*"

EXERCISE LV.

1. What will 50 lb. of butter cost, at 20¢ per pound?

2. What are 80 knives worth, at 25¢ each?

3. I bought 15 bu. of corn, at $33\frac{1}{3}$¢ per bu. How much did it cost me?

4. What will 64 yds. of cloth cost, at $6\frac{1}{4}$¢ per yd.?

5. I bought 24 quarts of strawberries, at $8\frac{1}{3}$¢ per qt. How much did I pay for them?

6. At $12\frac{1}{2}$¢ each, how many tin pails can be bought for $4?

7. At 25¢ each, how many melons can be bought for $5?

8. At 50¢ per bu., how many bushels of apples can be bought for $12?

9. At $33\frac{1}{3}$¢ each, how many books can be bought for $11?

10. 15 is what part of 75?

11. A bill of exchange on London costs $4.86 per £. How many £'s will the bill call for, if it cost $486?

NOTE.—The *English pound*, or *sovereign*, £, is worth about $4.86 in American gold.

12. Reduce 315 gallons to barrels (31.5 gal.=1 bbl.).

13. A block of marble 10 ft. long, 8 ft. wide, and 6 ft. high, is worth $420. Find the value of a block 5 ft. long, 2 ft. wide, and 2 ft. high.

14. A has $1000, which is 3 times as much as B has; C has 9 times as much as B. How much has C?

15. If 20 men build 24 miles of fence in 8 days, in how many days can 5 men build 8 miles?

16. Since 80° Réaumur is as hot as 100° Centigrade, how many degrees Centigrade will correspond to 240° Réaumur?

17. Reduce 160 ounces avoirdupois to pounds.

18. The sun passes over 15 degrees of longitude in 1 hour. In what time will it pass over 135 degrees?

19. John, who is 16 years old, is 4 times as old as James; Henry is 12 times as old as James: how old is Henry?

20. If 96 pounds of sea-water contains 8 pounds of salt, how many pounds of salt in 16 pounds of sea-water?

21. If the interest on \$500 for a certain time at a certain rate is \$25, what is the interest on \$125 for the same time at the same rate?

22. 15 times A's money is \$720. How much has B, who has 5 times as much as A?

23. The wages of 3 men for 2 days is \$9. Find the wages of 12 men for 8 days.

24. If 20 men in 15 days build 5 miles of fence, how many miles can 5 men build in 45 days?

80. Percentage Solutions.—In arithmetic, the term **Per Cent** means **hundredth** or **hundredths.** The sign, %, is used for the words *per cent.*

$$1\% = \tfrac{1}{100};$$
$$5\% = \tfrac{5}{100};$$
$$12\tfrac{1}{2}\% = \tfrac{12\frac{1}{2}}{100}; \text{ and}$$
$$100\% = \tfrac{100}{100}.$$

You should not think of percentage as something new in principle; it is only a new form of expressing *hundredths.*

PRINCIPLE: *100% of any number is all of it.*

To aid in shortening percentage solutions the following tables are given, and should be committed to memory:

TABLES.

2% = 1/50 of 100%	**12½% = 1/8 of 100%**
4% = 1/25 of 100%	**16⅔% = 1/6 of 100%**
5% = 1/20 of 100%	**20% = 1/5 of 100%**
6¼% = 1/16 of 100%	**25% = 1/4 of 100%**
8⅓% = 1/12 of 100%	**33⅓% = 1/3 of 100%**
10% = 1/10 of 100%	**50% = 1/2 of 100%**

100% = 50 × 2%	**100% = 8 × 12½%**
100% = 25 × 4%	**100% = 6 × 16⅔%**
100% = 20 × 5%	**100% = 5 × 20%**
100% = 16 × 6¼%	**100% = 4 × 25%**
100% = 12 × 8⅓%	**100% = 3 × 33⅓%**
100% = 10 × 10%	**100% = 2 × 50%**

EXAMPLES.

1. Find 12% of 630 pounds.

Solution: (1) 12% of 630 lb. = ( ) lb.? (Question.)
(2) 100% of 630 lb. = 630 lb. (Basis.)
$\frac{1}{100}$ of (2) = (3) 1% of 630 lb. = 6.3 lb.
12 × (3) = (4) 12% of 630 lb. = 75.6 lb., answer.

2. Mr. Jones has $1600, of which 25% is in the bank. How much has he in the bank?

Solution: (1) 25% of $1600 = ( )? (Question.)
(2) 100% of $1600 = $1600. (Basis.)
$\frac{1}{4}$ of (2) = (3) 25% of $1600 = $400, answer.

NOTE.—By knowing (from the table) that 25% is $\frac{1}{4}$ of 100%, we at once take $\frac{1}{4}$ of the basis, which gives the required answer without having to pass first to unity.

3. I sell goods for Mr. Carr, amounting to $400, on which I charge a commission of 2%. Find my commission.

Solution: (1) 2% of sales = $( )? (Question.)
(2) 100% of sales = $400. (Basis.)
$\frac{1}{50}$ of (2) = (3) 2% of sales = $8, answer.

NOTE.—Commission for selling is always some % of the selling price.

4. I bought goods for a merchant, amounting to $80. I charged 5% commission. Find my commission.

Solution: (1) 5% of purchase = $( )? (Question.)
(2) 100% of purchase = $80. (Basis.)
$\frac{1}{20}$ of (2) = (3) 5% of purchase = $4, answer.

NOTE.—Commission for buying is always some % of the purchase price.

5. An article cost \$4.20, and was sold at a profit of $16\frac{2}{3}\%$. Find the gain.

Solution: (1) $16\frac{2}{3}\%$ of cost = \$( )? (Question.)
(2) 100% of cost = \$4.20. (Basis.)
$\frac{1}{6}$ of (2) = (3) $16\frac{2}{3}\%$ of cost = \$.70, answer.

Note.—The profit or loss is always some % of the cost price.

6. I bought goods for \$300, and sold them at $33\frac{1}{3}\%$ profit. Find the selling price.

Solution: (1) $33\frac{1}{3}\%$ of cost = \$( )? (Question.)
(2) 100% of cost = \$300. (Basis.)
$\frac{1}{3}$ of (2) = (3) $33\frac{1}{3}\%$ of cost = \$100.
(4) \$300 + \$100 = \$400, answer.

Note.—The *profit* added to the *cost* gives the *selling* price.

7. I bought a horse for \$72, and sold him at a loss of $12\frac{1}{2}\%$. Find the selling price.

Solution: (1) $12\frac{1}{2}\%$ of cost = \$( )? (Question.)
(2) 100% of cost = \$72. (Basis.)
$\frac{1}{8}$ of (2) = (3) $12\frac{1}{2}\%$ of cost = \$9.
(4) \$72 − \$9 = \$63, answer.

Note.—The *loss* subtracted from the *cost* gives the selling price.

8. 17% of a number is 68. Find the number.

Solution: (1) 100% of number = ( )? (Question.)
(2) 17% of number = 68. (Basis.)
$\frac{1}{17}$ of (2) = (3) 1% of number = 4.
100 × (3) = (4) 100% of number = 400, answer.

Note.—When we cannot solve by a shorter plan, we can always go from what we have given to 1%; then from 1% to the required number of %.

9. A lawyer is paid 5% for collecting a debt. If he gets \$45, what is the amount of the debt?

Solution: (1) 100% of debt = \$( )? (Question.)
(2) 5% of debt = \$45. (Basis.)
20 × (2) = (3) 100% of debt = \$900, answer.

10. My gain on a certain article, which I sold at a profit of 25%, is \$.75. Find the cost price.

Solution: (1) 100% of cost = \$( )? (Question.)
(2) 25% of cost = \$.75. (Basis.)
$4\times(2)=(3)$ 100% of cost = \$3, answer.

11. My loss on a certain article, which I sold at a loss of $16\frac{2}{3}$%, is \$1.25. Find the cost price.

Solution: (1) 100% of cost = \$( )? (Question.)
(2) $16\frac{2}{3}$% of cost = \$1.25. (Basis.)
$6\times(2)=(3)$ 100% of cost = \$7.50, answer.

12. In one book-case I have 50 books; this is $12\frac{1}{2}$% of my library. How many books have I in my library?

Solution: (1) 100% of library = ( ) books? (Question.)
(2) $12\frac{1}{2}$% of library = 50 books. (Basis.)
$8\times(2)=(3)$ 100% of library = 400 books, answer.

13. 55 is 25% more than what number?

Solution: (1) 100% of No. = ( )? (Question.)
(2) 100% of No. + 25% of No. = 55. (Basis.)
$(2)=(3)$ 125% of No. = 55.
$\frac{1}{5}$ of $(3)=(4)$ 25% of No. = 11.
$4\times(4)=(5)$ 100% of No. = 44, answer.

14. An article was sold at 50% above cost for \$4.20. What did the article cost?

Solution: (1) 100% of cost = \$( )? (Question.)
(2) 100% of cost + 50% of cost = \$4.20. (Basis.)
$(2)=(3)$ 150% of cost = \$4.20.
$\frac{1}{3}$ of $(3)=(4)$ 50% of cost = \$1.40.
$2\times(4)=(5)$ 100% of cost = \$2.80, answer.

15. 70 is $12\frac{1}{2}$% less than what number?

Solution: (1) 100% of No. = ( )? (Question.)
(2) 100% of No. − $12\frac{1}{2}$% of No. = 70. (Basis.)
$(2)=(3)$ $87\frac{1}{2}$% of No. = 70.
$\frac{1}{7}$ of $(3)=(4)$ $12\frac{1}{2}$% of No. = 10.
$8\times(4)=(5)$ 100% of No. = 80, answer.

16. \$7 is what % of \$20?

Solution: (1) \$7 = ( )% of \$20? (Question.)
(2) \$20 = 100% of \$20. (Basis.)
$\frac{1}{20}$ of (2) = (3) \$1 = 5% of \$20.
$7 \times$ (3) = (4) \$7 = 35% of \$20, answer.

17. A lawyer gets \$35 commission for collecting a debt of \$500. What % commission does he charge?

Solution: (1) \$35 = ( )% of debt? (Question.)
(2) \$500 = 100% of debt. (Basis.)
$\frac{1}{500}$ of (2) = (3) \$1 = $\frac{1}{5}$% of debt.
$35 \times$ (3) = (4) \$35 = 7% of debt, answer.

18. I paid \$52 for a buggy, and sold it so as to gain \$13. Find the % of profit.

NOTE.—As *profit or loss* is some per cent of the *cost*, in this problem we really want to know how many % \$13 is of the cost, or \$52.

Solution: (1) \$13 = ( )% of cost? (Question.)
(2) \$52 = 100% of cost. (Basis.)
$\frac{1}{4}$ of (2) = (3) \$13 = 25% of cost, answer.

19. Find 8% of 15% of \$3500.

Solution: (1) 15% of \$3500 = \$( )? (Question.)
(2) 100% of 100% of \$3500 = \$3500. (Basis.)
$\frac{1}{100}$ of (2) = (3) 1% of \$3500 = \$35.
$15 \times$ (3) = (4) 15% of \$3500 = \$525.

(5) 8% of \$525 = \$( )? (Question.)
(6) 100% of \$525 = \$525. (Basis.)
$\frac{1}{100}$ of (6) = (7) 1% of \$525 = \$5.25.
$8 \times$ (7) = (8) 8% of \$525 = \$42, answer.

PLAN.—The plan of this solution is to find 15% of \$3500; then, 8% of that result. This gives 8% of 15% of \$3500.

Another form: (1) 8% of 15% of \$3500 = \$( )? (Question.)
(2) 100% of 100% of \$3500 = \$3500. (Basis.)
$\frac{1}{100}$ of (2) = (3) 1% of 100% of \$3500 = \$35.
$\frac{1}{100}$ of (3) = (4) 1% of 1% of \$3500 = \$.35.
$8 \times$ (4) = (5) 8% of 1% of \$3500 = \$2.80.
$15 \times$ (5) = (6) 8% of 15% of \$3500 = \$42, answer.

20. 75 is how many % more than 60?

Solution: (1) 75 = ( )% of 60? (Question.)
(2) 60 = 100% of 60. (Basis.)
$\frac{1}{60}$ of (2) = (3) 1 = $\frac{10}{6}$% of 60.
75 × (3) = (4) 75 = 125% of 60.
∴ 75 is 25% more than 60.

NOTE.— Since 100% of 60 is all of 60, 125% of 60 is 25% more than 60.

21. 40 is how many % less than 50?

Solution: (1) 40 = ( )% of 50? (Question.)
(2) 50 = 100% of 50. (Basis.)
$\frac{1}{50}$ of (2) = (3) 1 = 2% of 50.
40 × (3) = (4) 40 = 80% of 50.
∴ 40 is 20% less than 50.

22. A's money is 25% more than B's; then, B's money is how many % less than A's?

Solution: (1) 100% of B's money = ( )% of A's money? (Question.)
(2) 100% of B's money + 25% of B's money = 100% of A's money. (Basis.
(2) = (3) 125% of B's money = 100% of A's money.
$\frac{1}{125}$ of (3) = (4) 1% of B's money = $\frac{4}{5}$% of A's money.
100 × (4) = (5) 100% of B's money = 80% of A's money.
∴ B's money is 20% less than A's.

23. A's money is 20% less than B's; then, B's money is how many % more than A's?

Solution: (1) 100% of B's money = ( )% of A's money? (Question.)
(2) 100% of B's money − 20% of B's money = 100% of A's money. (Basis.)
(2) = (3) 80% of B's money = 100% of A's money.
$\frac{1}{4}$ of (3) = (4) 20% of B's money = 25% of A's money.
5 × (4) = (5) 100% of B's money = 125% of A's money.
∴ B's money is 25% more than A's.

EXERCISE LVI.

1. Find 6% of \$300.

2. Find $12\frac{1}{2}$% of \$4000.

3. Find 20% of $\frac{1}{2}$ of \$1600.

4. 8% of 30% of \$360 is how much?

5. Find 40% of 50% of \$$\frac{1}{2}$.

6. I owe A \$640; B, \$830; C, \$720; D, \$.25. How much will each get, if I can pay but 20%, or \$.20 on the \$1?

7. A certain tax levy is $1\frac{1}{2}$%. Find A's taxes, if his property is worth \$46420.

8. 9% of a debt is \$189.72. What is the debt?

9. \$3200 is 16% of what?

10. \$3200 is 40% of what?

11. \$75 is $33\frac{1}{3}$% more than what number?

12. \$80 is $16\frac{2}{3}$% less than what number?

13. \$250 is how many % more than \$200?

14. \$750 is how many % less than \$900?

15. A's money is $12\frac{1}{2}$% more than B's; then, B's money is how many % less than A's?

16. A's money is 25% less than B's; then, B's money is how many % more than A's?

17. A has a flock of 1800 sheep. He sells 15% to B, who sells 10% of his to C. After these sales how many has each?

18. I drew from a bank \$60, which was 15% of my deposit. What sum had I left in the bank?

19. B sells a pair of shoes at 10% less than list price; if list price is $33\frac{1}{3}$% above cost, and B sells them for \$1.80, find the cost.

20. What is the face value of Government bonds which pay, at 6%, an income of \$720 per annum?

21. 1 bushel is how many per cent more than 3 pecks?

22. 1 yard is how many per cent less than 1 meter?

NOTE.—36 inches = 1 yard; 39.37 inches = 1 meter.

23. 1 meter is how many per cent more than 1 yard?

24. 1 pint is what per cent of 1 gallon?

25. 5 months is what % of 9 months?

26. 20% of 50% of $600 is what per cent of 80% of 30% of $400?

Plan.—(1) Find 80% of 30% of $400; (2) find 20% of 50% of $600; (3) find what % the second result is of the first result.

Study the definitions and principles (omitting the formulas and relations), and solve by the method here given all the problems in Profit and Loss, Trade Discount, Commission, Stocks and Bonds, Taxes, Duties, and Insurance. (See Part II, pp. 251–280.)

2. THE PROPORTION METHOD.

81. Process.—The process of solving a problem by the *proportion method* consists in (1) stating a proportion according to the following principle, and (2) solving that proportion.

Principle: *In two equations of the same nature, and depending upon the same condition, the numerical ratio between the first members is equal to the numerical ratio between the second members.*

ILLUSTRATION.

Suppose, (a) *Cost of 1 book = $3,*

Then, 40 × (1) = (b) Cost of 40 books = $120.
30 × (1) = (c) Cost of 30 books = $90.
20 × (1) = (d) Cost of 20 books = $60.
15 × (1) = (e) Cost of 15 books = $45.
5 × (1) = (f) Cost of 5 books = $15.

Note.—The last five equations were all obtained from the first, and (1) are all of the *same nature*, having "*cost of books*" for their first members and "*$'s*" for their second members; (2) they all depend upon the *same condition—that 1 book costs $3.*

1. Compare (b) (d).

(b) Cost of 40 books = $120.
(d) Cost of 20 books = $60.
(1) 40 : 20 = 2.
(2) 120 : 60 = 2.
∴ 40 : 20 : : 120 : 60.

2. Compare (c) and (f).

(c) Cost of 30 books = \$90.
(f) Cost of 5 books = \$15.
(1) $30:5=6$.
(2) $90:15=6$.
$\therefore$ $15:5::90:15$.

3. Compare (e) and (f).

(e) Cost of 15 books = \$45.
(f) Cost of 5 books = \$15.
(1) $15:5=3$.
(2) $45:15=3$.
$\therefore$ $15:5::45:15$.

When the parts of a problem are stated for solution, the equations formed are of the same nature and depend upon the same condition; therefore, the above principle may be used in stating the proportion. After the proportion is stated it is solved by methods in Articles 68 and 69.

EXAMPLES.

1. If 10 books cost \$20, what will 35 books cost?

Solution: (1) Cost of 35 books = \$( )? (Question.)
(2) Cost of 10 books = \$20. (Basis.)
(3) $35:10::(\ ):20$? (Proportion.)
(4) $\dfrac{35\times \overset{2}{\cancel{20}}}{\cancel{10}}=70.$
$\therefore$ the required number is \$70.

2. If I can buy 15 hogs for \$75, how many can I buy for \$90?

Solution: (1) \$90 = cost of ( ) hogs? (Question.)
(2) \$75 = cost of 15 hogs. (Basis.)
(3) $90:75::(\ ):15$? (Proportion.)
(4) $\dfrac{\overset{18}{\cancel{90}}\times \cancel{15}}{\underset{\cancel{5}}{\cancel{75}}}=18$
$\therefore$ the required number is 18 hogs.

3. Reduce 8 bushels to pecks.

Solution: (1) 8 bu. = ( ) pk.? (Question.)
(2) 1 bu. = 4 pk. (Basis.)
(3) 8 : 1 : : ( ) : 4? (Proportion.)
(4) $\frac{8\times 4}{1}=32.$
∴ the required number is 32 pecks.

4. If 30 acres of land cost $450, how much was that per acre?

Solution: (1) Cost of 1 acre = $( )? (Question.)
(2) Cost of 30 acres = $450. (Basis.)
(3) 1 : 30 : : ( ) : 450? (Proportion.)
(4) $\frac{1\times \overset{15}{\cancel{450}}}{\cancel{30}}=15.$
∴ the required answer is $15.

5. If 50 shares of stock cost $4000, what will 78 shares cost?

Solution: (1) Price of 78 shares = $( )? (Question.)
(2) Price of 50 shares = $4000. (Basis.)
(3) 78 : 50 : : ( ) : 4000? (Proportion.)
(4) $\frac{78\times 4000}{50}=6240.$
∴ 78 shares will cost $6240.

6. A tract of land 40 rods long, 20 rods wide, contains 5 acres. What must be the length of a tract 60 rods wide to contain 30 acres?

1st Solution: (1) Length of a tract of 30 A., 60 rd. w. = ( ) rd.? (Question.)
(2) Length of a tract of 5 A., 20 rd. w. = 40 rd. (Basis.)
(3) $\frac{30}{60}:\frac{5}{20}::(\ ):40$? (Proportion.)
(4) $\frac{30\times 20\times 40}{60\times 5}=80.$
∴ the required length is 80 rods.

NOTE.—"$\frac{30}{60}$" and "$\frac{5}{20}$" do not in fact truly represent the numerical values of the two lengths considered in this problem. The "30" and

"5" are *acres*, while the "60" and "20" are *rods*. There are 160 square rods in an acre, and $\frac{30\times 160}{60}$ and $\frac{5\times 160}{20}$ would represent the true numerical length. But, in working out the proportion, one of these 160's would cancel the other, and they may therefore be omitted altogether.

It is not necessary in the proportion method that the blank term should always appear alone in the right member of the question, or in the right member at all for that matter; it may fall anywhere in the equation, but care must always be taken to arrange the *question and basis in the same order*. Number 6 may be solved as follows:

2d Solution: (1) Area of a tract ( ) rd. l., 60 rd. w. = 30 A.? (Question.)
(2) Area of a tract 40 rd. l., 20 rd. w. = 5 A. (Basis.)
(3) $(\ )\times 60 : 40\times 20 :: 30 : 5$? (Proportion.)
(4) $\frac{40\times 20\times 30}{60\times 5}=80.$
$\therefore$ the required length is 80 rods.

7. A tract of land 40 rods long, 20 rods wide, is worth $800. Find the width of a tract 50 rods long that is worth $1500.

Solution: (1) Value of a tract 50 rd. l., ( ) rd. w. = $1500? (Question.)
(2) Value of a tract 40 rd. l., 20 rd. w. = $800. (Basis.)
(3) $50\times(\ ) : 40\times 20 :: 1500 : 800$? (Proportion.)
(4) $\frac{40\times 20\times 1500}{50\times 800}=30.$
$\therefore$ the required width is 30 rods.

8. Goods that cost me $320, I sell for $416. How should I mark goods that cost me $40 to make a proportional profit in selling?

Solution: (1) Selling price of $40 worth of goods = $( )? (Question.)
(2) Selling price of $320 worth of goods = $416. (Basis.)
(3) $40 : 320 :: (\ ) : 416$? (Proportion.)
(4) $\frac{40\times 416}{320}=52.$
$\therefore$ the goods should be marked to sell for $52.

9. I sell 40 pounds of sugar, 16 ounces to the pound, for \$3.20. For how much should I sell 40 pounds, 15 ounces to the pound?

Solution: (1) Value of 40 lb., 15 oz. to the lb. = \$( )? (Question.)
(2) Value of 40 lb., 16 oz. to the lb. = \$3.20. (Basis.)
(3) $40 \times 15 : 40 \times 16 :: (\ ) : 3.2$? (Proportion.)
(4) $\frac{40 \times 15 \times 3.2}{40 \times 16} = 3.$
$\therefore$ the required value is \$3.

NOTE.—When the same factor occurs in each part of the problem, it has no effect upon the result. If the 40's were omitted in the above problem, would that make any change in the result?

10. A grocer sells a quantity of sugar by a false weight of 15 ounces to the pound for \$3.20. How much does he gain by the cheat?

Solution: (1) Cost of the number of lb. of 15 oz. each = \$( )? (Question.)
(2) Cost of same number of lb. of 16 oz. each = \$3.20. (Basis.)
(3) $15 : 16 :: (\ ) : 3.2$? (Proportion.)
(4) $\frac{15 \times 3.2}{16} = 3.$
$\therefore$ the true selling price is \$3, and his gain by cheating is 20¢.

NOTE.—The thief gets pay for true weight (16 oz. to the lb.). If he had delivered full weight, the sales would have been worth \$3.20; what is the amount worth which he does deliver?

11. A lawyer who collects for 5%, gets \$34.60 for collecting a debt. Find the amount of the debt.

Solution: (1) 100% of debt = \$( )? (Question.)
(2) 5% of debt = \$34.50. (Basis.)
(3) $100 : 5 :: (\ ) : 34.5$? (Proportion.)
(4) $\frac{100 \times 34.5}{5} = 690.$
$\therefore$ the amount of the debt is \$690.

12. A's property is assessed at \$3800. What is his tax at 96¢ on the \$100?

Solution: (1) Tax on \$3800 = \$( ) ? (Question.)
(2) Tax on \$100 = \$.96. (Basis.)
(3) 3800 : 100 : : ( ) : .96 ? (Proportion.)
(4) $\frac{3800 \times .96}{100} = 36.48.$
∴ A's tax is \$36.48.

13. B owns \$2500 of the capital stock of a \$200000 stock company. The company has \$14000 for distribution among its stockholders. How much does B get?

Solution: (1) The dividend on \$2500 = \$( ) ? (Question.)
(2) The dividend on \$200000 = \$14000. (Basis.)
(3) 2500 : 200000 : : ( ) : 14000 ? (Proportion.)
(4) $\frac{2500 \times 14000}{200000} = 175.$
∴ B's dividend is \$175.

14. A certain body of soldiers, standing 125 in rank and 60 in file, change the file to 75: what is the change in rank?

Solution:
(1) Length of a certain body of men 75 men w. = ( ) men ? (Question.)
(2) Length of same body of men 60 men w. = 125 men. (Basis.)
(3) $\frac{\text{No.}}{75} : \frac{\text{No.}}{60}$: : ( ) : 125 ? (Proportion.)
(4) $\frac{\text{No.} \times 125 \times 60}{75 \times \text{No.}} = 100.$
∴ the rank is changed to 100 men.

NOTE.—If there were two different numbers of men, the amount of each would have to be expressed; but as it is, the same "*No.*" (whatever its size) falls both *above* and *below*, and does not affect the result.

15. If 800 reams of paper are consumed in printing 38400 volumes of a 160-page book, octavo size, how many reams of paper will be required in printing 27000 volumes of a 320-page book, duodecimo size?

NOTE.—In making a book *octavo size* each sheet is folded into 8 leaves, making 16 pages; in a duodecimo size each sheet is folded into 12 leaves, making 24 pages.

Solution :

(1) Amt. of paper reqd. for 27000 vol. of 320 pp. each = ( ) reams of 24 pp. to the sheet ? (Question.)

(2) Amt. of paper reqd. for 38400 vol. of 160 pp. each = 800 reams of 16 pp. to the sheet. (Basis.)

(3) $27000 \times 320 : 38400 \times 160 :: (\ ) \times 24 : 800 \times 16$? (Proportion.)

(4) $\frac{27000 \times 320 \times 800 \times 16}{38400 \times 160 \times 24} = 750.$

∴ 750 reams will be required.

16. What rate is required for \$600 to produce \$192 interest in 4 years ?

Solution : (1) Int. on \$600 for 4 yr. at ( )% = \$192 ? (Question.)

(2) Int. on \$1 for 1 yr. at 1% = \$.01. (Basis.)

(3) $600 \times 4 \times (\ ) : 1 \times 1 \times 1 :: 192 : .01$? (Proportion.)

(4) $\frac{1 \times 1 \times 1 \times 192}{600 \times 4 \times .01} = \frac{19200}{2400} = 8.$

∴ the reqd. rate is 8%.

17. What principal will be required to bear \$225 at 9% in $3\frac{1}{3}$ years ?

Solution : (1) Int. on \$( ) for $3\frac{1}{3}$ yr. at 9% = \$225 ? (Question.)

(2) Int. on \$1 for 1 yr. at 1% = \$.01. (Basis.)

(3) $(\ ) \times 3\frac{1}{3} \times 9 : 1 \times 1 \times 1 :: 225 : .01$? (Proportion.)

(4) $\frac{1 \times 1 \times 1 \times 225}{3\frac{1}{3} \times 9 \times .01} = 750.$

∴ the required principal is \$750.

18. If 8 men mow 36 acres in 9 days of 9 hours each, how many men can mow 48 acres in 12 days of 12 hours each ?

Solution :

(1) Amt. mowed by ( ) men in 12 da. of 12 hr. each = 48 A. ? (Question.)

(2) Amt. mowed by 8 men in 9 da. of 9 hr. each = 36 A. (Basis.)

(3) $(\ ) \times 12 \times 12 : 8 \times 9 \times 9 :: 48 : 36$? (Proportion.)

(4) $\frac{8 \times 9 \times 9 \times 48}{12 \times 12 \times 36} = 6$

∴ the required answer is 6 men.

19. A garrison of men have food to last 9 months, giving each man 1 pound 2 ounces per day. What should be the daily allowance to make the same food last 1 year 8 months?

NOTE.—(1) 1 lb. 2 oz. = 18 oz.; 1 yr. 8 mo. = 20 mo.

Solution:

(1) $\left\{\begin{array}{c}\text{Food eaten by garrison}\\ \text{in 20 mo. at ( ) oz. each per da.}\end{array}\right\}$ = a certain amount? (Question.)

(2) $\left\{\begin{array}{c}\text{Food eaten by garrison}\\ \text{in 9 mo. at 18 oz. each per da.}\end{array}\right\}$ = same amount. (Basis.)

(3) $20\times(\ ):9\times18::\text{Amt.}:\text{Amt.}$? (Proportion.)

(4) $\dfrac{9\times18\times\text{amt.}}{20\times\text{amt.}}=8\frac{1}{10}$

∴ the required answer is $8\frac{1}{10}$ oz.

20. If 50 men in 10 days of 9 hours each, build a wall 250 yards long, 8 feet high, 3 feet thick, how many hours per day must 75 men work to build a wall 650 yards long, 9 feet high, 4 feet thick, in 30 days?

Solution: (1) $\left\{\begin{array}{l}\text{Amt. built by 75 men}\\ \text{working 30 da. of}\\ \text{( ) hr. each}\end{array}\right\}=\left\{\begin{array}{l}\text{A wall 650 yd. l.,}\\ \text{9 ft. h., 4 ft. th?}\end{array}\right.$ (Question.)

(2) $\left\{\begin{array}{l}\text{Amt. built by 50 men}\\ \text{working 10 da. of}\\ \text{9 hr. each}\end{array}\right\}=\left\{\begin{array}{l}\text{A wall 250 yd. l.,}\\ \text{8 ft. h., 3 ft. th.}\end{array}\right.$ (Basis.)

(3) $75\times30\times(\ ):50\times10\times9::650\times9\times4:250\times8\times3$? (Proportion.)

(4) $\dfrac{50\times10\times9\times650\times9\times4}{75\times30\times250\times8\times3}=7.8$

∴ the required answer is 7.8 hr.

EXERCISE LVII.

1. Find the cost of 150 bushels of apples at \$.65 per bushel.

2. If 12 tons of hay can be bought for \$36, how much must be paid for 50 tons?

3. A farmer sows 6 bushels of grain on $4\frac{4}{5}$ acres. At that rate, how many bushels will be needed to sow 3.6 acres?

4. If it requires 42 yd. of carpet $\frac{3}{4}$ yd. wide to cover a floor, how many yards of carpet 1 yd. wide will be needed?

5. In preparing a certain mixture, 12 gills of water were mixed with 7 gills of other ingredients. How many gills of other ingredients should be mixed with 42 gills of water?

6. 15% of a debt is $16.50. Find 80% of it.

7. A grocer uses false weight whereby $15\frac{1}{4}$ oz. is sold for a pound. What is the true value of groceries which he sells for $320?

8. If 20 yd. of cloth 1.5 yd. wide sell for $15, how wide must 51 yd. of the same kind of cloth be to sell for $42.50?

9. What are a servant's wages for 3 weeks 5 days, at $1.75 per week?

10. A body of soldiers are 42 in rank and 24 in file. If they were changed to 36 in rank, how many in file would they be?

11. An equatorial degree is 365000 feet. How many feet in 75° 24′, measured on the equator?

12. If a troy pound of English standard silver is worth £$3\frac{3}{10}$, what is 1 lb. av. worth?

13. If coffee which cost $225 is now worth $318.75, what was the cost of coffee now worth $1285.20?

14. A bin 7 ft. long, $2\frac{1}{2}$ ft. wide, 2 ft. high, contains 28 bu. of corn. How deep must one be which is 18 ft. long, $4\frac{4}{9}$ ft. wide, to contain 120 bushels?

15. If 150000 bricks are used in building a house whose walls average $1\frac{1}{2}$ ft. thick, 30 ft. high, and all together are 216 ft. long, how many bricks will be required to build a wall 2 ft. thick, 24 ft. high, and 324 ft. long?

16. It requires 800 reams of paper to print 28800 volumes of a duodecimo book of 320 pages to the volume. How many reams of paper will be required to print 24000 volumes of an octavo book of 520 pages to the volume?

17. The premium on a draft for $4320 is $21.60: how large a draft can I buy at a premium of $32.10?

18. The rate of premium as in No. 17. What will be the premium on a draft for $1234.20?

19. A property worth \$2540 rents for \$139.70: at that rate, what should property valued at \$4000 rent for?

20. At the rate in No. 19, what is the value of property which rents for \$192.70?

21. What is the tax on \$3450 worth of property at \$1.60 on the \$100?

22. If A charges 5¢ on the dollar for collecting a note and receives as his commission \$24.50, what is the face of the note?

23. A man failing is able to pay only \$65 on the \$100 of his debts. What does E get, if this man owes him \$824?

24. A stock company whose capital is \$4256000 is ready to distribute \$148960 among its stockholders. Smith gets a dividend of \$157.50: how much stock has he?

25. If 16 men build 18 rods of fence in 12 days, how many men can build 72 rods in 8 days?

26. If \$100 gain \$8 in 12 months, how many dollars will gain \$144 in 4 months?

27. If 6 men spend \$150 in 8 months, how much will 15 men spend in 20 months?

28. If 180 men in 6 days of 10 hours each dig a trench 200 yd. long, 3 yd. wide, 2 yd. deep, in how many days of 8 hr. each can 100 men dig a trench 180 yd. long, 4 yd. wide, 3 yd. deep?

29. If a regiment of 840 men has food for 60 days, how many days should the same food last a garrison of 1260 men?

30. If a pipe discharging 4 gallons per minute fill a cistern in 2 hours, in how many minutes may the cistern be filled by a pipe discharging 9 gallons per minute?

31. If the use of \$1500 for 3 yr. 8 mo. 25 da. is worth \$336.25, what is the use of \$100 for 1 yr. worth?

32. If 4 horses draw a car 9 miles per hour, how many miles per hour will an engine of 150 horse-power draw a train of 12 such cars, adding its own weight, 3 cars?

PART II.

I. STUDY OF NUMBERS.

A. DENOMINATE NUMBERS.

82. Definitions.—A **denominate number** represents quantity as composed of units of a particular *denomination* or *kind*. As,

50 bushels, 30 minutes, 8 pounds.

Most magnitudes may be expressed in units of two or more sizes. Thus, *time* may be expressed in seconds, minutes, hours, etc.

The process of determining the number of units of one size in a number expressed in units of another size, is called **reduction.** As,

(1) To find how many quarts in 10 gallons; or,
(2) To find how many gallons in 40 quarts.

When the change is from a larger (higher) to a smaller (lower) unit, as in (1), the reduction is **descending;** when from a smaller to a larger unit, as in (2), the reduction is **ascending.**

In order to change or reduce a number from units of one size to units of another, the relative size of these units must be known. The equation which states this relation is the *basis* of solution. These equations are given in the *tables*. The plan and process of solution are the same as learned in Part I.

There are two systems of measurement in use in the United States, *The English System* and *The French System.*

The English System, originated by the English, has come into common use among the people of the United States for all ordinary measurements and computations; except the English units of value, which are not used in the United States.

The French System, originated by the French, is used in the United States principally by scientists in making scientific measurements and computations.

I. ENGLISH SYSTEM.

83. Linear Measures.—A **Line** is that which has only one dimension, *length.* The linear measures, or units, are used in measuring heights, lengths, widths, distances, etc. There are two tables, the **Common Linear Table** and the **Surveyors' Linear Table.**

COMMON LINEAR TABLE.

1 yard (yd.)=3 feet (ft.).
1 ft.=12 inches (in.).

SURVEYORS' LINEAR TABLE.

1 mile (mi.)=80 chains (ch.).

$$1 \text{ chain} = \begin{cases} 4 \text{ rods (rd.), or} \\ 66 \text{ ft., or} \\ 100 \text{ links (l.).} \end{cases}$$

$$1 \text{ rod} = \begin{cases} 16\frac{1}{2} \text{ ft., or} \\ 25 \text{ l.} \end{cases}$$

NOTE 1.—A link is 7.92 inches long. It is seldom, if ever, necessary in practice to reduce links to inches or inches to links.

NOTE 2.—In ordinary land-surveying, surveyors use a chain 100 links, or 66 feet, long,—sometimes called *Gunter's Chain.* Civil engineers generally use a "*steel tape,*" 100 feet long.

EXAMPLES.

1. Reduce 5 yd. 2 ft. 9 in. to inches.

PLAN.—(1) Reduce the yards to feet, and add the *2 ft.* (2) Reduce the feet to inches, and add the *9 in.* Each step contains one problem and one addition.

Solution: (1) 5 yd. = ( ) ft.? (Question.)
(2) 1 yd. = 3 ft. (Basis.)
5 × (2) = (3) 5 yd. = 15 ft.
(4) 15 ft. + 2 ft. = 17 ft.

(5) 17 ft. = ( ) in.? (Question.)
(6) 1 ft. = 12 in. (Basis.)
17 × (6) = (7) 17 ft. = 204 in.
(8) 204 in. + 9 in. = 213 in., answer.

2. Reduce 2 mi. 20 ch. 74 l. to links.

NOTE.—If, in the judgment of the teacher, the solution is sufficiently clear to the pupil without stating the "*Question*" each time, the "*Question*" may be omitted. Thus,

Solution: (1) 1 mi. = 80 ch. (Basis.)
2 × (1) = (2) 2 mi. = 160 ch.
(3) 160 ch. + 20 ch. = 180 ch.

(4) 1 ch. = 100 l. (Basis.)
180 × (4) = (5) 180 ch. = 18000 l.
(6) 18000 l. + 74 l. = 18074 l., answer.

3. Prepare the surveyors' linear table for use in reduction ascending.

NOTE.—In reduction descending, the order of the process is from higher to lower units, and the *larger* unit in each equation is on the left. In reduction ascending, this order is reversed, and the smaller unit is placed on the left. Thus,

$$1 \text{ in.} = \frac{1}{7.92}\text{l.} = \frac{25}{198}\text{l.}$$

$$1 \text{ l.} = \begin{cases} \frac{1}{100} \text{ ch.} \\ \frac{1}{25} \text{ rd.} \end{cases}$$

$1 \text{ ft.} = \begin{cases} \frac{2}{33} \text{ rd.} \\ \frac{1}{66} \text{ ch.} \end{cases}$

1 rd. $= \frac{1}{4}$ ch.

1 ch. $= \frac{1}{80}$ mi.

4. Reduce 24348 l. to higher units.

Solution: (1) 1 l. $= \frac{1}{100}$ ch. (Basis.)
24348 × (1) = (2) 24348 l. $= \frac{24348}{100}$ ch. = 243 ch. + 48 l.
(3) 1 ch. $= \frac{1}{80}$ mi. (Basis.)
243 × (3) = (4) 243 ch. = 3 mi. + 3 ch.
∴ 24348 l. = 3 mi. 3 ch. 48 l.

Another Solution: (1) 1 l. $= \frac{1}{25}$ rd. (Basis.)
24348 × (1) = (2) 24348 l. = 973 rd. + 23 l.
(3) 1 rd. $= \frac{1}{4}$ ch. (Basis.)
973 × (3) = (4) 973 rd. = 243 ch. + 1 rd.
(5) 1 ch. $= \frac{1}{80}$ mi. (Basis.)
243 × (5) = (6) 243 ch. = 3 mi. + 3 ch.
∴ 24348 l. = 3 mi. 3 ch. 1 rd. 23 l.

NOTE.—In the first solution, equation (2), $\frac{24348}{100}$ ch. is simplified by dividing the 24348 by 100. The quotient is 243 ch., the remainder is $\frac{48}{100}$ ch. But $\frac{1}{100}$ ch. is 1 l., therefore $\frac{48}{100}$ ch. = 48 l. Thus, the quotient represents the number of *chains*, and the remainder the number of *links*.

5. Reduce 444 rd. to miles.

Solution: (1) 1 rd. $= \frac{1}{4}$ ch. (Basis.)
444 × (1) = (2) 444 rd. = 111 ch.
(3) 1 ch. $= \frac{1}{80}$ mi. (Basis.)
111 × (3) = (4) 111 ch. $= 1\frac{31}{80}$ mi., or, 1.3875 mi., answer.

NOTE.—In reducing to "*higher units*," the remainders are left as *integers*, and the quotient only is reduced higher. In reducing to a *particular denomination*, as in No. 5, the *exact quotient* is obtained each time and all carried to the required denomination. In reducing fractions to "*lower units*," when the result is a mixed number, leave the integral part, and reduce only the fraction lower.

6. Reduce $\frac{2}{3}$ yd. $\frac{1}{4}$ ft. to inches.

Solution: (1) 1 yd. = 3 ft. (Basis.)
$\frac{2}{3}$ of (1) = (2) $\frac{2}{3}$ yd. = 2 ft.
(3) 2 ft. + $\frac{1}{4}$ ft. = $2\frac{1}{4}$ ft.

(4) 1 ft. = 12 in. (Basis.)
$2\frac{1}{4}\times$(4) = (5) $2\frac{1}{4}$ ft. = 27 in., answer.

7. Reduce $\frac{5}{16}$ ft. to the fraction of a mile.

Solution: (1) 1 ft. = $\frac{1}{66}$ ch. (Basis.)
$\frac{5}{16}$ of (1) = (2) $\frac{5}{16}$ ft. = $\frac{5}{1056}$ ch.

(3) 1 ch. = $\frac{1}{80}$ mi. (Basis.)
$\frac{5}{1056}$ of (3) = (4) $\frac{5}{1056}$ ch. = $\frac{1}{16896}$ mi., answer.

8. Reduce .32 rd. to the decimal of a mile.

Solution: (1) 1 rd. = $\frac{1}{4}$ ch. = .25 ch.
.32 of (1) = (2) .32 rd. = .08 ch.

(3) 1 ch. = $\frac{1}{80}$ mi. = .0125 mi.
.08 of (3) = (4) .08 ch. = .001 mi., answer.

NOTE.—When a decimal result is required, it may be obtained as above by expressing all fractions as decimals; or the solution may be expressed in common fractions and the result only reduced to a decimal.

9. Find in integral units the value of $\frac{2}{9}$ mi. minus $\frac{5}{11}$ rd.

Solution: (1) 1 mi. = 80 ch. (Basis.)
$\frac{2}{9}$ of (1) = (2) $\frac{2}{9}$ mi. = $17\frac{7}{9}$ ch.

(3) 1 ch. = 4 rd. (Basis.)
$\frac{7}{9}$ of (3) = (4) $\frac{7}{9}$ ch. = $\frac{28}{9}$ rd.
(5) $\frac{28}{9}$ rd. − $\frac{5}{11}$ rd. = $\frac{308}{99}$ rd. − $\frac{45}{99}$ rd. = $\frac{263}{99}$ rd. = $2\frac{73}{99}$ rd.

(6) 1 rd. = $16\frac{1}{2}$ ft. = $\frac{33}{2}$ ft. (Basis.)
$\frac{73}{99}$ of (6) = (7) $\frac{73}{99}$ rd. = $\frac{73}{99}\times\frac{33}{2}$ ft. = $12\frac{1}{6}$ ft.

(8) 1 ft. = 12 in. (Basis.)
$\frac{1}{6}$ of (8) = (9) $\frac{1}{6}$ ft. = 2 in.
$\therefore$ $\frac{2}{9}$ mi. − $\frac{5}{11}$ rd. = 17 ch. 2 rd. 12 ft. 2 in.

10. **Reduce $\frac{1}{2}$ ft. $\frac{3}{4}$ rd. $\frac{5}{8}$ ch. to the fraction of a mile.**

Solution: (1) 1 ft. $= \frac{2}{33}$ rd. (Basis.)
(2) $\frac{1}{2}$ ft. $= \frac{1}{33}$ rd.
(3) $\frac{1}{33}$ rd. $+ \frac{3}{4}$ rd. $= \frac{4}{132}$ rd. $+ \frac{99}{132}$ rd. $= \frac{103}{132}$ rd.

(4) 1 rd. $= \frac{1}{4}$ ch. (Basis.)
(5) $\frac{103}{132}$ rd. $= \frac{103}{528}$ ch.
(6) $\frac{103}{528}$ ch. $+ \frac{5}{8}$ ch. $= \frac{103}{528}$ ch. $+ \frac{330}{528}$ ch. $= \frac{433}{528}$ ch.

(7) 1 ch. $= \frac{1}{80}$ mi. (Basis.)
(8) $\frac{433}{528}$ ch. $= \frac{433}{42240}$ mi., answer.

NOTE.—Notice that number 10 may be stated in the form of an example of addition. Thus, $\frac{1}{2}$ ft. $+ \frac{3}{4}$ yd. $+ \frac{5}{8}$ ch. $=$ ( ) mi.?

EXERCISE LVIII.

1. Reduce 1 mi. to rods.
2. Reduce 1 mi. to feet.
3. Reduce 7 mi. 240 rd. to inches.
4. Reduce 3 mi. 79 ch. to links.
5. How many yards in 1 mile?

PLAN.—(1) Reduce 1 mi. to ft., (2) reduce the ft. to yd.

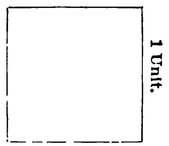
6. Reduce $\frac{11}{12}$ mi to integral units.
7. Reduce $\frac{17}{55}$ mi. $\frac{5}{33}$ ch. $\frac{5}{6}$ rd. $\frac{1}{4}$ ft. to inches.
8. Reduce $\frac{14}{33}$ mi. $\frac{5}{11}$ ch. $\frac{3}{4}$ rd. to integral units.
9. Reduce .35 mi. 7.3 ch. to inches.
10. Reduce .37 mi. 5.7 ch. to links.
11. Reduce 340000 l. to miles.
12. Reduce 248000 in. to higher units.
13. Reduce 12480 ft. to higher units.
14. Reduce 25 ft. to the fraction of a chain.
15. Reduce 33 ft. to the fraction of a mile.

16. Reduce 5.5 ft. to the decimal of a mile.
17. Reduce $\frac{7}{11}$ l. to the fraction of a mile.
18. Reduce 1 l. 1 ft. 1 rd. 1 ch. to the fraction of a mile.
19. $\frac{3}{11}$ ch. $-\frac{5}{12}$ rd. $=($ $)$ mi.?

PLAN.—(1) Reduce the ch. to rd. and subtract; (2) reduce the result to mi.

20. $\frac{7}{12}$ mi. $+\frac{3}{25}$ ch. $+\frac{3}{5}$ ft. $=($ $)$ in.?
21. $\frac{5}{9}$ mi. $-\frac{2}{15}$ ch. $+\frac{4}{5}$ rd. -2 ft. $=($ $)$ in.?
22. 5 ch. -3.5 rd. $+4\frac{3}{4}$ ft. -11 in. $=($ $)$ mi.?

84. Surface Measures.—A **Surface** is that which has only two dimensions, *length* and *width*. The number of square units in a surface is its **Area.** A **Square Unit** is the amount of a surface 1 unit long and 1 unit wide.



Square Unit.

There are two tables of surface measures, corresponding to the two tables of linear measures. The one is the **Common Square Measures,** used to measure the area of such surfaces as floors, ceilings, carpets, cloth, and the like.

TABLE.

1 sq. yd. = 9 sq. ft.
1 sq. ft. = 144 sq. in.

The other table is the **Surveyors' Square Measures,** used in measuring the area of land.

TABLE.

1 township (Tp.) = 36 sq. mi., or Sections (Sec.).
1 sq. mi. = 640 acres (A.).
1 A. = $\begin{cases} 160 \text{ sq. rd.} \\ 10 \text{ sq. ch.} \end{cases}$
1 sq. ch. = 16 sq. rd.
1 sq. rd. = 272¼ sq. ft.

EXERCISE LIX.

1. How many acres in a section?

2. How many acres in a township?

3. How many square rods in a square mile?

4. Reduce 12 sq. yd. 7 sq. ft. to square inches.

5. Reduce 25 sq. yd. 5 sq. ft. to square inches.

6. Fill the following blanks, preparing the table for work in reduction ascending:

1 sq. ft. = $\frac{4}{1089}$ sq. rd.
1 sq. rd. = $\begin{cases} (\ \ ) \text{ sq. ch.?} \\ (\ \ ) \text{ A.?} \end{cases}$
1 sq. ch. = ( ) A.?
1 A. = ( ) Sec.?
1 Sec. = ( ) Tp.?

Also,

1 sq. in. = ( ) sq. ft.?
1 sq. ft. = ( ) sq. yd.?

7. Reduce 2592 sq. in. to sq. yd.

8. Reduce 32000 A. to sections.

9. How many townships in 648 sq. mi.?

10. How many square miles in 231200 sq. rd.?

11. How many sq. ch. in 2 Sec.?

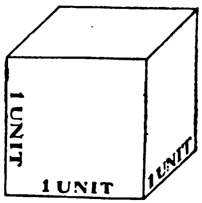
12. Reduce 7 sq. mi. 120 A. 150 sq. rd. to square rods.

13. Reduce 202480 sq. rd. to higher units.
14. Reduce $\frac{5}{8}$ A. to lower units.
15. Reduce .625 sq. rd. to lower units.
16. Reduce $\frac{7}{24}$ sq. yd. to sq. inches.
17. Reduce 10 sq. ft. to the fraction of a sq. yd.
18. Reduce $\frac{81}{125}$ sq. in. to the decimal of a sq. yd.
19. $\frac{3}{5}$ sq. mi. + .5 A. + 120 sq. rd. = ( ) sq. rd.?
20. 12.5 sq. ch. + $6\frac{3}{4}$ sq. rd. = ( ) sq. mi.?
21. .6 sq. yd. = ( ) A.?

PLAN.—(1) Reduce sq. yd. to sq. ft.; (2) reduce sq. ft. to sq. rd.; (3) reduce sq. rd. to A.

22. $\frac{7}{12}$ A. − $\frac{5}{8}$ sq. rd. + 240 sq. ft. = ( ) sq. in.?
23. $\frac{7}{12}$ A. − $\frac{5}{8}$ sq. rd. + 240 sq. ft. = ( ) Sec.?
24. 5 sq. mi. − 20.4 A. + $\frac{7}{12}$ sq. ch. = ( ) Tp.?

85. Solid Measures.—A **Solid** is that which has three dimensions, *length*, *width*, and *thickness*. The number of cubic units in a solid is its **Volume**. A **Cubic Unit** is the amount of a solid 1 unit long, 1 unit wide and 1 unit thick.



Cubic Unit.

The *Cubic Measures* are used in measuring volumes of solids, and the capacities of bins, tanks, and the like.

TABLE.

1 cu. yd. = 27 cu. ft.
1 cu. ft. = 1728 cu. in.

EXERCISE LX.

1. Reduce 5 cu. yd. 15 cu. ft. to cu. in.

2. A cord of wood is 8 ft. long, 4 ft. wide, and 4 ft. high. How many cubic feet does it contain ?

Basis: Vol. of a solid 1 ft. l., 1 ft. w., 1 ft. h. = 1 cu. ft.

3. How many cubic inches in a cord ?

4. Reduce 5 cords (C.) to cubic ft.

5. 288 cu. in. is what part of a cubic yd.?

NOTE.—Prepare the table for reduction ascending before solving No. 5.

6. 12 cu. ft. 6 cu. in. is what part of 1 C.?

7. Reduce $\frac{5}{12}$ cu. ft. to cu. yd.

8. $\frac{5}{9}$ cu. yd. + $24\frac{3}{4}$ cu. ft. = ( ) cu. in.?

9. A perch of masonry is $16\frac{1}{2}$ ft. long, $1\frac{1}{2}$ ft. thick, and 1 ft. high. How many cubic feet does it contain ?

10. $\frac{5}{18}$ cu. ft. − $\frac{7}{36}$ cu. in. = ( ) cu. yd.?

86. Measures of Capacity.—The extent of room or space within a vessel is called its **Capacity.** Measures of capacity are classed as *Dry Measures* and *Liquid Measures.* The dry measures are used to measure quantities of grain, fruits, seeds, and the like.

TABLE.

1 bushel (bu.) = 4 pecks (pk.).
1 pk. = 8 quarts (qt.).
1 qt. = 2 pints (pt.).

NOTE.—1 bushel contains 2150.4+ cu. in. Fruits, vegetables, seeds and the like are bought and sold by the *bushel;* but the amount of the bushel is often expressed by weight rather than size. (See p. 161.)

EXERCISE LXI.

1. Reduce 20 bu. to pints.

2. Reduce 2 bu. 2 pk. to quarts.

3. Reduce 7 bu. 3 pk. 7 qt. to pints.

4. Reduce 9 bu. 1 pk. 4 qt. 1 pt. to pints.

5. Reduce 768 pt. to bushels.

6. Reduce 679 pt. to higher units.

7. Reduce $\frac{3}{5}$ bu. to pints.

8. Reduce $\frac{19}{64}$ bu. to lower units.

9. $\frac{3}{4}$ bu. $+\frac{3}{4}$ pk. $+\frac{3}{4}$ qt. $+\frac{3}{4}$ pt. $=(\quad)$ pt.?

10. $\frac{3}{4}$ bu. $+\frac{3}{4}$ pk. $+\frac{3}{4}$ qt. $+\frac{3}{4}$ pt. $=(\quad)$ bu.?

11. 5 bu. $-\frac{3}{4}$ pk. $+\frac{3}{8}$ qt. $-1\frac{1}{2}$ pt. $=(\quad)$ pt.?

12. $\frac{1}{2}$ bu. $-\frac{3}{4}$ pk. $+.375$ qt. $-1\frac{1}{2}$ pt. $=(\quad)$ bu.?

13. Find the value of 4 bu. 3 pk. of apples at 20¢ per peck.

14. A grocer buys $2\frac{1}{2}$ bu. of cherries; he then sells 1 bu. 3 pk. of them; again, he buys 1 bu. 1 pk. 3 qt. of cherries; finally he sells all the cherries he has at 6¢ per quart. Find the value of the last sale.

There are two tables of Liquid Measures in use, (1) the *Common Liquid Measures*, used in measuring such liquids as water and milk, and (2) the *Apothecaries' Liquid Measures*, used by apothecaries in measuring liquid medicines:

COMMON TABLE.

1 hogshead (hhd.) = 2 barrels (bbl.).
1 bbl. = 31½ gallons (gal.).
1 gal. = 4 qt.
1 qt. = 2 pt.
1 pt. = 4 gills (gi.).

NOTE.—1 gallon contains 231 cu. in. The *dry measure* quart contains 9.45 cu. in. more than the *liquid quart*.

APOTHECARIES' TABLE.

1 gallon (*Cong.)=8 pints (*O.)
1 O.=16 fluid ounces (f℥).
1 f℥=8 fluid drams (fʒ).
1 fʒ=60 minims (m.).

EXERCISE LXII.

1. Reduce 1 hhd. to gills.
2. Reduce 2 hhd. 1 bbl. 21 gal. to quarts.
3. Reduce ½ gal. ½ qt. to gills.
4. Reduce 1008 gi. to barrels.
5. Reduce 1008 pt. to higher units.
6. Reduce 5 Cong. to minims.
7. Reduce 5 O. 2 f℥ 3 fʒ to minims.
8. Reduce 1260 m. to higher units.
9. A physician wishes to prepare a 5% solution of carbolic acid. How much water and how much acid must be used to make 2 fluid ounces ?

NOTE.—5% of the 2 f℥ is carbolic acid, and the rest is water.

10. What will 1 bbl. of syrup sell for, at 30¢ per gallon?

87. Measures of Mass.—The **Mass** of a body is the amount of matter it contains. The **Weight** is the measure of the attraction between that body and the earth. *Weights* are used to measure the *mass* of a body.

NOTE.—To be accurate in comparing the masses of bodies by their weights, the weights must be taken at the same *altitude*, in the same *latitude*, and under the *same conditions*. But for ordinary purposes, this accuracy is not observed.

* The Latin words are *Congius* and *Octavus*.

There are three tables of weights: (1) The table of **Avoirdupois Weights,** used in weighing all ordinary articles, such as groceries, meats, live stock, etc.

TABLE.

1 ton. (T.)=20 hundredweight (cwt.).
1 cwt.=100 pounds (lb.).
1 lb.=16 ounces (oz.).

(2) The table of **Troy Weights,** used in *jewelry stores,* and at *mints,* when the Government weighs gold and silver for making money.

TABLE.

1 pound (lb.)=12 ounces (oz.).
1 oz.=20 pennyweights (pwt.).
1 pwt.=24 grains (gr.).

(3) The table of **Apothecaries' Weights,** used by druggists in filling prescriptions.

TABLE.

1 pound (lb.=12 ounces (℥).
1 ℥=8 drams (ʒ).
1 ʒ=3 scruples (℈)
1 ℈=20 grains (gr.).

NOTE.—The grain, ounce, and pound of the troy weights are identical with the grain, ounce, and pound, respectively, of the apothecaries' weights. The troy or apothecaries' pound contains 5760 gr., but the avoirdupois pound contains 7000 gr.

Grain and vegetables are often bought and sold by *weight-*

bushels instead of the measured bushels. In the majority of States the bushel is of the following weight:

TABLE.

1 bu. of wheat=60 lb.
1 bu. of corn=56 lb.
1 bu. of oats=32 lb.
1 bu. of Irish potatoes=60 lb.
1 bu. of sweet potatoes=55 lb.

NOTE.—The *pounds* of the above table are *avoirdupois pounds.* These equivalents are not the same in all States, and the teacher should teach the pupil the table corrected for his own State.

EXERCISE LXIII.

1. Reduce 1 T. 3 cwt. 4 lb. to ounces.

2. Reduce 7000 oz. to higher units.

3. Reduce 1 lb. 4℥ 5ʒ 2℈ 17 gr. to grains.

4. Reduce 6743 gr. troy to higher units.

5. Reduce 6743 gr. apothecaries' to higher units.

6. Reduce 1 lb. avoirdupois to highest integral units of the troy table.

7. Reduce 2 lb. apothecaries' to highest integral units of the avoirdupois table.

8. A man sells 9600 lb. of Irish potatoes at 70¢ per bu. How much do the potatoes bring?

9. A silver dollar weighs 17 pwt. 4½ gr. What is the weight of $80 in silver?

10. A druggist buys lactopeptine at $8 per lb. and sells it at 12½¢ per ʒ. What is his per cent of gain?

11. How many bales of 70 lb. each in 14 T. 17 cwt. 50 lb. of hay?

12. A man sold 3 loads of corn of 2352 lb. each. How many bushels did he sell?

88. Measures of Time.—These measures are used in giving the time of day, in Longitude and Time problems, and in computing the interest on notes and bills.

TABLE.

1 year (yr.)=12 months (mo.).
1 mo.=30 days (da.).
1 da.=24 hours (hr.).
1 hr.=60 minutes (min.).
1 min.=60 seconds (sec.).

Calendar Months:

January, 31 days.	July, 31 days.
*February, 28 days.	August, 31 days.
March, 31 days.	September, 30 days.
April, 30 days.	October, 31 days.
May, 31 days.	November, 30 days.
June, 30 days.	December, 31 days.

NOTE.—The above data give three different lengths to the year. (1) From the table, 1 yr. = 12 mo. = 360 da. This may be called the common *interest year*. The custom in the majority of the States is to use 360 da. for a year. (2) By counting the number of days of the calendar months, the length of the year is found to be 365 days. This is called the *common year*. (3) *Leap Year.*—The average length of a year (a complete revolution of the earth around the sun), expressed in mean solar time (sun time) is 365 days, 5 hours, 48 minutes, 47.8 seconds. The *Julian Calendar* called the year 365¼ days. For ordinary purposes it is best to use whole days; so the ¼ day each year amounted in 4 years to 1 day. In this way, the Julian Calendar had 3 *common years* of 365 days each, then 1 *leap year* of 366 days. The extra day was added to February. This calendar, by counting 365¼ days for a year, used 11 minutes 12.2 seconds too much. The Gregorian Calendar corrected this error by omitting the *leap year* on centennial years, except those which are multiples of 400. The act of Parlia-

* February has 29 days in *leap* years.

ment which changed from the Julian Calendar (old style) to the Gregorian Calendar (new style) dropped 11 days out of the month of September, 1752, making September 3d, *old style*, September 14th, *new style*. The civilized nations, except Russia, have adopted the Gregorian Calendar.

Remember that, when the number of a year is divisible by 4, it is a leap year; except a centennial year, whose number must be divisible by 400 in order to be a leap year.

There are 52 weeks 1 day in a common year, and 52 weeks 2 days in a leap year. A week has 7 days, named Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday.

EXERCISE LXIV.

1. Reduce 365 da. 5 hr. 48 min. 47.8 sec. to seconds.

2. Reduce 31556936 sec. to higher units.

3. Name the months in order and give the number of days in each.

4. Learn the number of each month, as January, 1st; September, 9th; June, 6th; etc.

5. Find which of the following are leap years: 1756, 1835, 1800, 1892, 1876, 1900, 1904, 1906, 1952, 2000.

6. Find the number of days from June 10, 1895, to March 17, 1896.

7. Find the exact time from 10 o'clock A. M., Wednesday, June 12, 1901, to noon the following Monday.

8. Find the exact time 9170 minutes after 3 P. M., Saturday, June 22, 1901.

9. When was a note due if it was dated June 12, 1901, and due in 93 days?

10. When was a note due if it was dated December 10, 1879, and due in 125 days?

89. Measures of Value.—The money of a country is its circulating medium and adopted standard of value. The following is the table of English units of value, or denominations of money.

TABLE.

1 pound (£)=20 shillings (s.).
1 s.=12 pence* (d.).
1 d.=4 farthings (far.).

NOTE.—The value of the English pound (or sovereign) in United States money is $4.8665.

EXERCISE LXV.

1. Reduce £5 to pence.
2. Reduce 7543 far. to £.
3. Reduce 8424 far. to higher units.
4. A bill for £200 12 s. is worth how many dollars?
5. A bill for $4300 is worth how much in English money?

90. Review and Rapid Drill Work.—Heretofore you have been required to consider each step in the process of reduction of denominate numbers as a problem, and to solve it as such.

Being now familiar with the nature of reduction, you should hereafter give your attention to speed and accuracy in obtaining the result.

DIRECTIONS.

FOR REDUCTION DESCENDING: *Multiply by the number of units of the next lower denomination which makes one unit of the denomination to be reduced.*

FOR REDUCTION ASCENDING: *Divide by the number of units of the given denomination, which makes one unit of the next higher denomination.*

NOTE.—Use the corresponding abstract numbers in this work.

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* *The singular of pence is penny.*

EXERCISE LXVI.

Reduce, orally—

1. 5 bu. to pk.
2. 6 qt. to pt.
3. 8 pk. to pt.
4. 14 pt. to qt.
5. 48 pt. to pk.
6. 128 qt. to bu.

7. 48 in. to ft.
8. 20 ch. to l.
9. 1 mi. to rd.
10. 300 l. to ch.
11. 198 ft. to ch.
12. 72 in. to yd.

13. 12 sq. yd. to sq. ft.
14. 3 sq. ft. to sq. in.
15. 1 mi. to sq. ch.
16. 20 A. to sq. rd.
17. 70 sq. ch. to A.
18. 480 sq. rd. to A.

19. 2 cu. yd. to cu. ft.
20. 1 cu. ft. to cu. in.
21. 1 bu. to cu. in.
22. 1 gal. to cu. in.
23. 462 cu. in. to gal.
24. 54 cu. ft. to cu. yd.

25. 1 day to min.
26. 1 hr. to sec.
27. 1 yr. to mo.
28. 50 min. to hr.
29. 6 hr. to da.
30. 3 mo. to yr.

31. 3 lb. av. to oz.
32. 3 lb. troy to oz.
33. 8 ℥ to ℈.
34. 4 pwt. to gr.
35. 5 ʒ to gr.
36. 128 oz. av. to lb.
37. 120 oz. troy to lb.
38. 96 ℥ to lb.
39. 24 ℈ to ℥.
40. 120 m. to fʒ.

EXAMPLES.

1. Reduce 5 bu. to pints.

Process: $5 \times 4 \times 8 \times 2 = 320$.
$\therefore$ 5 bu. = 320 pt.

Explanation: (1) Multiplying by 4 reduces to pecks; by 8, to quarts; by 2, to pints. Therefore the required number

is the continued product of 5, 4, 8, and 2, or 320. This multiplying should all be done mentally, and no more figures should be used than are given above.

2. Reduce 7 cwt. 5 lb. 4 oz. to ounces.

Process :

A.

(1) $100 \times 7 + 5 = 705$.

(2) $16 \times 705 + 4 = 11284$,

or,

B.

$$\begin{array}{r} 7 \\ \underline{100} \\ 705 \\ \underline{16} \\ 4230 \\ 705 \\ \underline{4} \\ 11284 \end{array}$$

$\therefore$ 7 cwt. 5 lb. 4 oz. = 11284 oz.

Explanation : If the pupil can multiply by 16 (and he ought to), use form *A ;* if not, use form *B*. In *B*, the 5 is added to the product of 7 and 100 before writing the result.

3. Reduce 1512 pt. to hogsheads.

Process :

A.

$$\begin{array}{r} 2\underline{)1512} \\ 4\underline{)756} \\ 189 \\ \underline{2} \\ 63)378(6 \div 2 = 3 \\ \underline{378} \end{array}$$

or,

B.

$$\frac{\overset{\overset{\overset{3}{\cancel{189}}}{\cancel{756}}}{\cancel{1512}} \times \cancel{2}}{\cancel{2} \times \cancel{4} \times \cancel{63} \times \cancel{2}} = 3.$$

$\therefore$ 1512 pt. = 3 hhd.

Explanation : Form *A* is a much-used form. Multiplying by 2 and dividing by 63 is the same as dividing by $31\frac{1}{2}$. The author has some preference for form *B*, when all or nearly all of the divisors are small.

4. Reduce 37560 hr. to years.

Process :

$$\frac{\overset{\overset{313}{\cancel{3130}}}{\cancel{37560}}}{\underset{3}{\cancel{24}} \times \cancel{30} \times \cancel{12}} = \frac{313}{72} = 4\tfrac{25}{72}.$$

$\therefore$ 37560 hr. = $4\frac{25}{72}$ yr.

5. Reduce 435 pt. (dry) to higher units.

Process:

$$\begin{array}{r@{}l} 2)\underline{435} & \\ 8)\underline{217} & +1 \\ 4)\underline{27} & +1 \\ 6 & +3 \end{array}$$

$\therefore$ 435 pt. = 6 bu. 3 pk. 1 qt. 1 pt.

6. Express in integral units $\frac{7}{120}$ hhd.

Process: (1) $\dfrac{7 \times \cancel{2}}{\underset{60}{\cancel{120}}} = \frac{7}{60}$.

(2) $\dfrac{7 \times \overset{21}{\cancel{63}}}{\underset{20}{\cancel{60}} \times 2} = \frac{147}{40} = 3\frac{27}{40}$.

(3) $\dfrac{27 \times \cancel{4}}{\underset{10}{\cancel{40}}} = \frac{27}{10} = 2\frac{7}{10}$.

(4) $\dfrac{7 \times \cancel{2}}{\underset{5}{\cancel{10}}} = 1\frac{2}{5}$.

$\therefore$ $\frac{7}{120}$ hhd. = 3 gal. 2 qt. $1\frac{2}{5}$ pt.

7. Reduce .175 bu. to lower integral units.

Process:

$$\begin{array}{r} .175 \\ \underline{4} \\ .7\cancel{0}\cancel{0} \\ \underline{8} \\ 5.6 \\ \underline{2} \\ 1.2 \end{array}$$

$\therefore$ .175 bu. = 5 qt. 1.2 pt.

8. Reduce 8.75 lb. troy to lb. av.

Process: $\dfrac{8.75 \times 5760}{7000} = 7\frac{1}{5}$.

$\therefore$ 8.75 lb. troy = $7\frac{1}{5}$ lb. av.

PLAN.—(1) Multiplying by 5760 reduces to gr., and (2) dividing by 7000 reduces to lb. av.

EXERCISE LXVII.

Reduce:

1. 2 mi. 20 ch. to inches.
2. 3 sq. mi. 5 A. to sq. rd.
3. 7 cu. yd. 5 cu. ft. to cu. in.
4. 7 bu. 5 qt. to pt.
5. $5\frac{1}{2}$ gal. 1 pt. to gi.
6. 5 O. 1 f℥ to m.
7. 5 T. 4 cwt. to oz.
8. 1 T. 6 oz. to oz.
9. $3\frac{1}{2}$ lb. troy to gr.
10. 2 ℥ 4 ʒ 2 ℈ to gr.
11. 5 yr. 4 mo. 17 da. to da.
12. £7 11 d. to far.

Reduce to higher units:

13. 143759 in.
14. 73498 cu. in.
15. 4375 gr. troy.
16. 3470 A.
17. 840 pt. (liquid).
18. 842 pt. (dry).
19. 546304 min.
20. 4444 oz. av.

Reduce to lowest denomination of the table:

21. $\frac{3}{16}$ yd.
22. .05 yr.
23. $\frac{7}{11}$ rd.
24. .34 hr.
25. $\frac{3}{4}$ bu. $\frac{3}{8}$ pk. 3 qt.
26. .5 sq. rd.
27. $\frac{1}{2}$ cu. yd. + .7 cu. ft.
28. .6 lb. + $\frac{3}{4}$ pwt.

Reduce to highest denomination of the table:

29. $5\frac{5}{16}$ gr. troy.
30. 5 pt. (dry).
31. 1008 sq. ch.
32. 549. cu. in.
33. 1 ʒ 2 ℈.
34. 120 da.
35. .75 qt. (liquid).
36. 7 pk. + .8 qt.

Reduce:

37. 5 lb. av. to lb. troy.
38. 7 lb. apoth. to lb. av.
39. 60 bu. to cu. in.
40. 456 gal. to cu. in.
41. 340 bu. to gal.
42. 576 gal. to bu.

2. FRENCH SYSTEM.

91. Linear Measures.

TABLE.

1 Myriameter (Mm.)=10 Kilometers (Km.).
1 Km.=10 Hectometers (Hm.).
1 Hm.=10 Dekameters (Dm.).
1 Dm.=10 meters (m.).
1 m.=10 decimeters (dm.).
1 dm.=10 centimeters (cm.).
1 cm.=10 millimeters (mm.).

NOTE.—*1 meter = 39.37 inches.* By means of this equivalent, any number expressed in units of this table may be expressed in units of the English linear table, and *vice versa.*

Reduction descending and ascending may be performed here just as in the English System; but, owing to the fact that we are now dealing with a decimal system—a system in which 10 units of one denomination make one of the next higher—the process of reduction may be much shortened. The shorter processes are developed in the following examples:

EXAMPLES.

1. Reduce 5 Hm. 4 Dm. 2 m. to m.

Process: 5 Hm. 4 Dm. 2m. = 542 m., result.

NOTE.—When there are no denominations omitted between the highest denomination given and the denomination required, the figures written in order side by side will be the abstract number corresponding to the required result.

2. Reduce 7 Mm. 9 Km. 4 Dm. to Dm.

Process: 7 Mm. 9 Km. 4 Dm. = 7904 Dm., result.

NOTE.—Put a *0* in place of the denomination omitted.

3. Reduce 5 Km. to dm.

Process: 5 Km. = 50000 dm., result.

NOTE.—Put a *0* in place of each denomination omitted, from Km. to dm. inclusive.

4. Reduce 34579 dm. to higher units.

Process : 34579 dm. = 3 Km. 4 Hm. 5 Dm. 7 m. 9 dm., result.

NOTE.—This is just the reverse of the process used in reduction descending. It is plain, that by dividing successively by 10's, the successive remainders would be 9, 7, 5, etc., and that the result would be just as given above.

5. Reduce 1605 mm. to higher units.

Process : 1605 mm. = 1 m. 6 dm. 5 mm., result.

NOTE.—The *0 cm.* is omitted in the result.

6. Reduce 5.046 Dm. to cm.

Process : 5.046 Dm. = 5046 cm., result.

NOTE.—To reduce from Dm. to cm., we must multiply by 10 three times. This is done by moving the decimal point three places to the right.

7. Reduce 54603.25 m. to Hm.

Process : 54603.25 m. = 546.0325 Hm.

NOTE.—To divide by 10 twice, move the decimal point two places to the left.

8. Reduce 3 Hm. 6 Dm. to yards.

Process : (1) 3 Hm. 6 Dm. = 360 m.

$$(2)\ \frac{360 \times 39.37}{36} = 393.7.$$

$\therefore$ 3 Hm. 6 Dm. = 393.7 yd.

NOTE.—Always reduce to *meters*, and then to *inches*.

9. Reduce 218 yd. 2 ft. 2 in. to Dm.

Process : (1) 218 yd. 2 ft. 2 in. = 7874 in.

$$(2)\ \frac{7874}{39.37 \times 10} = 20.$$

$\therefore$ 218 yd. 2 ft. 2 in. = 20 Dm.

NOTE.—Always reduce to *inches*, and then to *meters*.

EXERCISE LXVIII.

Reduce :

1. 5 Mm. to m.

2. 9 Km. 8 Dm. to dm.

3. 7 Hm. to mm.

4. 5 Mm. 7 Hm. to cm.

5. 8 m. 5 dm. to mm.

6. 43675 dm. to Km.

7. 4367.3 m. to Hm.

8. 4735 cm. to higher units.

9. 54843 m. to higher units.

10. 4.003 Dm. to mm.

11. 7 Mm. 8 Km. 6 Hm. 5 Dm. 4 m. to mm.
12. 3 Mm. 8 Hm. 7 Dm. 5 dm. 4 cm. to mm.
13. 36 Mm. to miles.
14. 236220 ft. to Dm.
15. 540.0632 Dm. to inches.

92. Surface Measures.

TABLE.

1 sq. Mm.=100 sq. Km.
1 sq. Km.=100 sq. Hm.
1 sq. Hm.=100 sq. Dm.
1 sq. Dm.=100 sq. m.
1 sq. m.=100 sq. dm.
1 sq. dm.=100 sq. cm.
1 sq. cm.=100 sq. mm.

NOTE.—A square Dekameter is also called an *are*. The *are* is the unit of land measure. A square meter is equal to 39.37×39.37 sq. in., or about 1550 sq. in.

EXERCISE LXIX.

Reduce :

1. 15 sq. Mm. to sq. m.
2. 75 sq. Km. to sq. dm.
3. 4 sq. Dm. to sq. mm.
4. 540000 sq. cm. to sq. m.
5. 89506000 sq. dm. to ares.
6. 48675056 sq. Dm. to higher units.
7. 520 sq. m. to sq. in.
8. 25 sq. Dm. 12 sq. m. to sq. in.
9. 17 sq. Km. to sq. yd.
10. 520 sq. rd. to sq. m.
11. 740 A. to ares.
12. 1 sq. mi. to sq. Hm.

93. Solid Measures.

TABLE.

1 cu. Mm. = 1000 cu. Km.
1 cu. Km. = 1000 cu. Hm.
1 cu. Hm. = 1000 cu. Dm.
1 cu. Dm. = 1000 cu. m.
1 cu. m. = 1000 cu. dm.
1 cu. dm. = 1000 cu. cm.
1 cu. cm. = 1000 cu. mm.

NOTE.—A cubic meter is also called a *stere*. The *stere* is used in measuring wood. A cubic meter is equal to $39.37 \times 39.37 \times 39.37$ cu. in., or about 61023.3779+ cu. in.

EXERCISE LXX.

1. 75 cu. Km. to cu. m.
2. 748 cu. Dm. to cu. cm.
3. 10 cu. m. to cu. ft.
4. 4856000000 cu. dm. to cu. Dm.
5. 8000000000 cu. m. to cu. Km.
6. 182769534 cu. ft. to cu. Dm.
7. How many cords of wood in a pile containing 500 cu. m.?
8. How many steres of wood in a pile containing 500 cu. yd.?

94. Measures of Capacity.

TABLE.

1 Myrialiter (Ml.) = 10 Kiloliters (Kl.).
1 Kl. = 10 Hectoliters (Hl.).
1 Hl. = 10 Dekaliters (Dl.).
1 Dl. = 10 liters (l.).
1 l = 10 deciliters (dl.).
1 dl. = 10 centiliters (cl.).
1 cl. = 10 milliliters (ml.).

NOTE.—The *liter* is equal in volume to a *cubic decimeter* (about 1.05 liquid quarts, or .9 dry quart).

EXERCISE LXXI.

Reduce:

1. 5 Kl. to l.	*7.* 40000 cl. to Hl.
2. 9 Ml. to Dl.	*8.* 45675 dl. to higher units.
3. 7 Hl. 5 l. to dl.	*9.* 7043.08 cl. to Dl.
4. 4 Kl. 4 Hl. to cl.	*10.* 48 l. to qt.
5. 5 Dl. 8 l. to qt.	*11.* 25 gal. to l.
6. 7 Hl. to gal.	*12.* 75 bu. to Dl.

95. Measures of Mass.

TABLE.

1 Tonneau (T.)=10 quintals (Q.).
1 Q.=10 Myriagrams (Mg.).
1 Mg.=10 Kilograms (Kg.).
1 Kg.=10 Hektograms (Hg.).
1 Hg.=10 Dekagrams (Dg.).
1 Dg.=10 grams (g.).
1 g.=10 decigrams (dg.).
1 dg.=10 centigrams (cg.).
1 cg.=10 milligrams (mg.).

NOTE.—The *gram* is the weight of a *cubic centimeter* of distilled water (about 15.42 grains troy).

EXERCISE LXXII.

Reduce:

1. 6 T. to g.	*7.* 240 dg. to g.
2. 9 Q. to Dg.	*8.* 4380 cg. to Dg.
3. 7 Mg. to dg.	*9.* 5863 cg. to higher units.
4. 5 Dg. to mg.	*10.* 840 dg. to gr.
5. 50 Mg. to lb. troy.	*11.* 123.36 lb. troy to Mg.
6. 50 Mg. to lb. av.	*12.* 123.36 lb. av. to Mg.

96. Measures of Value.

TABLE.

1 franc (fr.) = 10 decimes (d.).
1 d. = 10 centimes (c.).
1 c. = 10 millimes (m.).

NOTE.—The *franc* is equal in value to 19.3¢.

EXERCISE LXXIII.

Reduce:

1. 5 fr. to m.
2. 15 fr. 7d. to m.
3. 50 fr. to d.
4. 240 c. to fr.
5. 4834 m. to higher units.
6. $5.79 to fr.
7. 45 fr. to $'s.
8. 3800 c. to ¢.
9. 5790 m. to mills.
10. 3860 dimes to d.

3. COMPOUND DENOMINATE NUMBERS.

97. Addition.

EXAMPLES.

1. Add 1 bu. 2 pk. 4 qt. 1 pt.; 5 bu. 1 pk. 2 qt.

Process:

bu.	pk.	qt.	pt.	
1	2	4	1	
5	1	2	0	
6	3	6	1	, result.

Explanation: (1) Write the names of the successive units of the table used, putting them in horizontal order, with the largest to the left.

(2) Write the addends below—each part in its place, as indicated by the table above.

(3) Add as we have done heretofore, placing the sum obtained by adding each column below. *Result, 6 bu. 3 pk. 6 qt. 1 pt.*

2. Add 2 gal. 3 qt. 1 pt. 3 gi.; 7 gal. 1 pt. 3 gi.; 8 gal. 2 qt. 3 gi.

Process:

gal.	qt.	pt.	gi.	
2	3	1	3	
7	0	1	3	
8	2	0	3	
18	3	0	1	, result.

Explanation: (1) 3 gi. + 3 gi. + 3 gi. = 9 gi. But 9 gi. = 2 pt. 1 gi. Write the *1* below, and carry the *2 pt.*

(2) 2 pt. (carried) + 1 pt. + 1 pt. = 4 pt. But 4 pt. = 2 qt. Write *0*

below, and carry the *2 qt.* We use the *0* to show that there are no pints left.

(3) 2 qt. (carried) + 2 qt. + 3 qt. = 7 qt. = 1 gal. 3 qt. Write the *3 qt.* below, and carry the *1 gal.*

(4) 1 gal. (carried) + 8 gal. + 7 gal. + 2 gal. = 18 gal. Write 18 below. *Result, 18 gal. 3 qt. 1 gi.*

NOTE.—Teacher should point out the similarity between this and *Addition of Simple Numbers.*

3. Add 5 yd. 2 ft. 8 in.; 7 yd. 2 ft. 9 in.; 17 yd. 2 ft. 11 in.

Process:

yd.	ft.	in.
5	2	8
7	2	9
17	2	11
31	2	4, result.

Explanation: (1) Sum 28 in. = 2 ft. 4 in. Write the *4* below, and carry the *2*.

(2) Sum 8 ft. = 2 yd. 2 ft. Write the *2* (ft.) below, and carry the *2* (yd.).

(3) Sum 31 yd. Write the *31* below. *Result, 31 yd. 2 ft. 4 in.*

EXERCISE LXXIV.

Add:

1.

hr.	min.	sec.
7	18	36
9	35	40
11	43	34

2.

yr.	mo.	da.
50	8	12
46	5	25
2	9	14

3.

yd.	ft.	in.
9	1	11
8	0	3
11	2	7

4.

cu. yd.	cu. ft.	cu. in.
120	17	540
242	9	1048
483	21	1436
368	0	421

5.

sq. yd.	sq. ft.	sq. in.
12	0	73
46	8	96
52	5	100
37	7	77

6.

T.	cwt.	lb.
7	5	40
18	12	72
50	15	00
	18	84

7.

lb.	℥.	ʒ.	℈.	gr.
5	7	7	2	14
11	11	5	1	18
7	9	0	2	15

8.

hhd.	bbl.	gal.	qt.	pt.
2	1	25	3	1
4	0	30	2	1
5	1	11	1	0

98. Subtraction.

EXAMPLES.

1. From 15 T. 6 cwt. 12 lb. 5 oz. take 6 T. 10 cwt. 8 lb. 14 oz.

Process:

T.	cwt.	lb.	oz.
15	6	12	5
6	10	8	14
8	16	3	7, result.

Explanation: (1) 5 is smaller than 14. We must take 1 lb. from the 12 lb. and add it to the 5 oz. 1 lb.+5 oz.=21 oz. 21−14=7. Write the *7* below.

(2) There are only 11 lb. left in the minuend. (Why?) 11−8=3. Write the *3* below.

(3) 10 is larger than 6. (What shall we do?) 1 T.+6 cwt. =26 cwt. 26−10=16. Write the *16* below.

(4) There are 14 T. left in the minuend. (Why?) 14−6 =8. Write the *8* below. *Result, 8 T. 16 cwt. 3 lb. 7 oz.*

NOTE.—Teacher should point out the similarity between this and *Subtraction of Simple Numbers.*

2. A note dated July 10, 1897, was paid September 23, 1899. How long was the time between these dates?

Process:

yr.	mo.	da.
1899	9	23
1897	7	10
2	2	13, result.

Explanation: Write the latest date above. Write the number of the month: September, *9th* month; July, *7th* month. *The pupil should learn the number of each month in the year.*

3. When I looked at my watch last it was 18 min. after 9 o'clock, but it is now 12 min. after 11 o'clock. How long has it been since I looked at my watch?

Form:

hr.	min.
11	12
9	18
1	54, result.

NOTE.—18 minutes after 9 o'clock is 9 hr. 18 min. (from noon or midnight).

EXERCISE LXXV.

1. From 13 bu. 1 pk. take 8 bu. 2 pk.

2. From 2 cwt. 38 lb. take 1 cwt. 49 lb.

3. From 2 T. 5 cwt. 41 lb. take 1 T. 10 cwt. 74 lb.

4. From 5 mi. 65 ch. 1 rd. take 3 mi. 78 ch. 3 rd.

5. From 50 yr. 3 mo. 12 hr. 50 min. 10 sec. take 24 yr. 6 da. 5 min.

6. From £12 3 s. 9 d. 2 far. take £6 17 s. 11 d. 3 far.

7. From 25 cu. yd. 16 cu. ft. 127 cu. in. take 7 cu. yd. 26 cu. ft. 240 cu. in.

8. Find the time from Jan. 11, 1895, to June 1, 1901.

99. Multiplication.

EXAMPLES.

1. Multiply 3 bu. 1 pk. 5 qt. 1 pt. by 7.

Process:

bu.	pk.	qt.	pt.
3	1	5	1
			7
23	3	6	1, result.

Explanation: (1) 7×1 pt. = 7 pt. = 3 qt. 1 pt. Write the *1* below, and carry the *3*.

(2) 7×5 qt. = 35 qt. 35 qt. + 3 qt. (carried) = 38 qt. = 4 pk. 6 qt. Write the *6* below, and carry the *4*.

(3) 7×1 pk = 7 pk. 7 pk. + 4 pk. = 11 pk. = 2 bu. 3 pk. Write the *3* below, and carry the *2*.

(4) 7×3 bu. = 21 bu. 21 bu. + 2 bu. = 23 bu. Write the *23* below. ***Result**, 23 bu. 3 pk. 6 qt. 1 pt.*

NOTE.—The teacher should point out the similarity between this and *Multiplication of Simple Numbers*.

2. Multiply 27 yd. 2 ft. 9 in. by 9.

Process:

yd.	ft.	in.
27	2	9
		9
251	0	9, result.

Explanation: (1) 9×9 = 81. 81 in. = 6 ft. 9 in. Write the *9* below, and carry the *6*.

(2) 9×2 = 18. 18 ft. + 6 ft. = 24 ft. = 8 yd. Write *0* below, and carry the *8*.

(3) 9×27 = 243. 243 yd. + 8 yd. = 251 yd. Write the *251* below. ***Result**, 251 yd. 9 in.*

3. Multiply 5 T. 8 cwt. 54 lb. 6 oz. by 8.

Process:

T.	cwt.	lb.	oz.
5	8	54	6
			8
43	8	35	0, result.

NOTE.—Have the pupil explain this example.

EXERCISE LXXVI.

Multiply:

1. 12 yd. 1 ft. 10 in. by 8.
2. 5 mi. 43 ch. 51 ft. 9 in. by 11.
3. 17 sq. yd. 7 sq. ft. 7 sq. in. by 7.
4. 3 Tp. 21 Sec. 25 A. 9 sq. ch. by 25.
5. 11 cu. yd. 16 cu. ft. 128 cu. in. by 20.
6. 24 bu. 3 qt. by 24.
7. 4 gal. 3 qt. 1 pt. 2 gi. by 8.
8. 6 Cong. 5 O. 7 f℥ 3 fʒ 35 m. by 12.
9. 7 T. 13 cwt. 46 lb. 10 oz. by 120.
10. 5 lb. 4 oz. 18 pwt. 22 gr. by 9.
11. 8 lb. 9℥ 7ʒ 2℈ 5 gr. by 24.
12. 5 yr. 7 mo. 3 da. by 40.

100. Division.

EXAMPLES.

1. Divide 27 bu. 3 pk. 7 qt. by 6.

Process:

	bu.	pk.	qt.	pt.	
6)	27	3	3	0	
	4	2	4	1,	result.

Explanation: (1) 27 ÷ 6 = 4, remainder 3. Write the *4* below, and reduce the 3 bu. to pecks.

(2) 3 bu. + 3 pk. = 15 pk. 15 ÷ 6 = 2, remainder 3. Write the *2* below, and reduce the 3 pk. to quarts.

(3) 3 pk. + 3 qt. = 27 qt. 27 ÷ 6 = 4, remainder 3. Write the *4* below, and reduce the 3 qt. to pints. 3 qt. = 6 pt. 6 ÷ 6 = 1. Write the *1* below. ***Result, 4 bu. 2 pk. 4 qt. 1 pt.***

NOTE.—Teacher should point out the similarity between this and *Division of Simple Numbers.*

2. Divide 112 T. 16 cwt. 59 lb. by 7.

Process:

	T.	cwt.	lb.	
7)	112	16	59	
	16	2	37,	result.

NOTE.—The pupil should explain this example.

EXERCISE LXVII.

Divide:

1. 490 bu. 2 pk. 4 qt. by 100.
2. 161 yd. 1 ft. 6 in. by 17.
3. 3 yr. 11 mo. 4 da. by 14.
4. 65 gal. 2 qt. 1 pt. by 7.
5. 3 lb. apoth. by 120.
6. 1 Tp. 5 sq. mi. 430 A. by 21.

B. INVOLUTION.

101. Definitions.—A composite number that can be formed by using the same number as a factor a number of times is called a **Power** of the number used; and the number used is called the **Root** of the power. The *degree* of a power is the number of times the root is used in producing the power. Thus,

$$3\times3=9.$$

9 is the second power of 3.

A small figure placed just above and to the right of a number indicates the degree of the power to which the number is to be raised, and is called an *Exponent*. The number itself is the *first* power; the number used twice as a factor gives the *second power* or *square;* the number used three times as a factor gives the *third power* or *cube;* used four times gives the *fourth power;* five times, the *fifth power*, and so on. Thus,

$3^1=3$ **(First power.)**
$3^2=3\times3=9$ **(Second power or square.)**
$3^3=3\times3\times3=27$ **(Third power or cube.)**
$3^4=3\times3\times3\times3=81$ **(Fourth power.)**
$3^5=3\times3\times3\times3\times3=243$ **(Fifth power.)**
$3^6=3\times3\times3\times3\times3\times3=729$ **(Sixth power.)**

Involution is the process of finding a required power of a given root.

102. Process. Since by the definition a power is a *product*, the process of finding a power is *multiplication*.

EXAMPLES.

1. $5^4 = (\ \ )$?

Process: $5^4 = 5 \times 5 \times 5 \times 5 = 625$, result.

2. $534^2 = (\ \ )$?

Process:

$$\begin{array}{r} 534 \\ 534 \\ \hline 2136 \\ 1602 \\ 2670 \\ \hline 285156 \end{array}$$

, result.

3. $(\frac{5}{12})^3 = (\ \ )$?

Process: $(\frac{5}{12})^3 = \frac{5}{12} \times \frac{5}{12} \times \frac{5}{12} = \frac{125}{1728}$, result.

NOTE.—To raise a common fraction to a given power, both the numerator and denominator must be raised to that power.

4. $.043^3 = (\ \ )$?

Process:

$$\begin{array}{r} .043 \\ .043 \\ \hline 129 \\ 172 \\ \hline .001849 \\ .043 \\ \hline 5547 \\ 7396 \\ \hline .000079507 \end{array}$$

, result.

Explanation: Proceed as in multiplication of decimals.

EXERCISE LXXVIII.

Commit to memory the powers:

1. 1^2; 2^2; 3^2; 4^2; 5^2; 6^2; 7^2; 8^2; 9^2; 10^2.

2. 1^3; 2^3; 3^3; 4^3; 5^3; 6^3; 7^3; 8^3; 9^3; 10^3.

Raise the following numbers to the powers indicated:

1. 17^3.	*5.* $(\frac{5}{9})^6$.	*9.* 47^2.
2. 127^2.	*6.* $(\frac{12}{29})^3$.	*10.* $.031^3$.
3. 37^4.	*7.* $(37\frac{1}{2})^3$.	*11.* 5.06^2.
4. 13^5.	*8.* $(245\frac{1}{3})^2$.	*12.* $(.14\frac{5}{8})^3$.

103. Another Method. The multiplication may be performed by separating the root into two numbers, and multiplying each part of the multiplicand by each part of the multiplier.

EXAMPLES.

1. $5^2=(3+2)^2=(\quad)$?

Process:

$$\begin{array}{r} 3+2 \\ \underline{3+2} \\ 6+4 \\ \underline{9+\ 6} \\ 9+12+4=25, \text{ result.} \end{array}$$

2. $7^2=(5+2)^2=(\quad)$?

Process:

$$\begin{array}{r} 5+2 \\ \underline{5+2} \\ 10+2^2 \\ \underline{5^2+10} \\ 5^2+2\times10+2^2=25+20+4=49, \text{ result.} \end{array}$$

These examples are illustrative of the following principle:

PRINCIPLE: *The square of the sum of two numbers is equal to the square of the first plus twice the product of the first times the second plus the square of the second.*

By referring to the two numbers as 1st and 2d, this principle may be stated in an equation as follows:

$$(\mathbf{1st}+\mathbf{2d})^2=\mathbf{1st}^2+2\times\mathbf{1st}\times\mathbf{2d}+\mathbf{2d}^2. \text{ (Commit.)}$$

3. $6^2=(\quad)$?

Process: $6^2=(4+2)^2=4^2+2\times4\times2+2^2=16+16+4=36$; or,
$6^2=(3+3)^2=3^2+2\times3\times3+3^2=9+18+9=36$; or,
$6^2=(1+5)^2=1^2+2\times1\times5+5^2=1+10+25=36$, result.

NOTE.—A number may be separated into any two of its addends and this principle will apply.

4. $(17)^2=(\quad)$?

Process: $17^2=(10+7)^2=10^2+2\times10\times7+7^2=100+140+49=289$, result.

5. $5^3=(\quad)$?

Process:

$$\begin{array}{r}
2+\ 3 \\
2+\ 3 \\
\hline
6+\ 9 \\
4+\ 6 \\
\hline
4+12+\ 9 \\
2+\ 3 \\
\hline
12+36+27 \\
8+24+18 \\
\hline
8+36+54+27
\end{array}=125\text{, result.}$$

6. $9^3=(\quad)$?

Process:

$$\begin{array}{r}
5+4 \\
5+4 \\
\hline
5\times4+4^2 \\
5^2+5\times4 \\
\hline
5^2+2\times5\times4+4^2 \\
5+4 \\
\hline
5^2\times4+2\times5\times4^2+4^3 \\
5^3+2\times5^2\times4+\quad5\times4^2 \\
\hline
5^3+3\times5^2\times4+3\times5\times4^2+4^3
\end{array}=729\text{, result.}$$

NOTE.—The teacher should carefully explain each step in this multiplication.

This example illustrates the following principle:

PRINCIPLE: *The cube of the sum of two numbers is equal to the cube of the first plus three times the square of the first times the second plus three times the first times the square of the second plus the cube of the second; or,*

$$(1st+2d)^3=1st^3+3\times 1st^2\times 2d+3\times 1st\times 2d^2+2d^3.$$
(Commit.)

7. $23^3=(\ \ )$?

Process: $23^3=(20+3)^3=20^3+3\times 20^2\times 3+3\times 20\times 3^2+3^3=$ $8000+3600+540+27=12167$, result.

NOTE.—These two principles have been here presented, not so much for their practical value in *involution*, as to prepare for their use in *evolution*—extracting square root and cube root.

EXERCISE LXXIX.

1. $11^2=(7+4)^2=(\ \ )$?
2. $15^2=(8+7)^2=(\ \ )$?
3. $13^2=(5+8)^2=(\ \ )$?
4. $21^2=(20+1)^2=(\ \ )$?
5. $34^2=(30+4)^2=(\ \ )$?
6. $8^3=(3+5)^3=(\ \ )$?
7. $12^3=(10+2)^3=(\ \ )$?
8. $14^3=(10+4)^3=(\ \ )$?
9. $23^3=(20+3)^3=(\ \ )$?
10. $46^3=(40+6)^3=(\ \ )$?

C. EVOLUTION.

104. Definition.—Evolution is the process of finding a required root of a given power.

The number itself is the *first root;* one of the two equal factors which compose a number is its *second* or *square root;* one of the three equal factors which compose a number is its *third* or *cube root;* one of the four equal factors which compose a number is its *fourth root;* and so on.

The required root of a given number may be indicated in either of two ways:

(1) By the use of the *radical sign*, $\sqrt{\ }$, with a small figure or figures placed between the two parts to indicate the degree of the root. Thus,

$\sqrt[2]{121}$, the square root of 121.

$\sqrt[3]{1728}$, the cube root of 1728.

$\sqrt[4]{256}$, the fourth root of 256.

3. $6^2 = (\quad)$?

Process: $6^2=(4+2)^2=4^2+2\times4\times2+2^2=16+16+4=36$; or,
$6^2=(3+3)^2=3^2+2\times3\times3+3^2=9+18+9=36$; or,
$6^2=(1+5)^2=1^2+2\times1\times5+5^2=1+10+25=36$, result.

NOTE.—A number may be separated into any two of its addends and this principle will apply.

4. $(17)^2=(\quad)$?

Process: $17^2=(10+7)^2=10^2+2\times10\times7+7^2=100+140+49=289$, result.

5. $5^3=(\quad)$?

Process:

$$\begin{array}{rrrl}
 & & 2+ & 3 \\
 & & 2+ & 3 \\
\cline{3-4}
 & & 6+ & 9 \\
 & 4+ & 6 & \\
\cline{2-4}
 & 4+ & 12+ & 9 \\
 & & 2+ & 3 \\
\cline{2-4}
 & 12+ & 36+ & 27 \\
8+ & 24+ & 18 & \\
\hline
8+ & 36+ & 54+ & 27=125\text{, result.}
\end{array}$$

6. $9^3=(\quad)$?

Process:

$$\begin{array}{rrrl}
 & & 5+ & 4 \\
 & & 5+ & 4 \\
\cline{3-4}
 & & 5\times4+ & 4^2 \\
 & 5^2+ & 5\times4 & \\
\cline{2-4}
 & 5^2+ & 2\times5\times4+ & 4^2 \\
 & & 5+ & 4 \\
\cline{2-4}
 & 5^2\times4+ & 2\times5\times4^2+ & 4^3 \\
5^3+ & 2\times5^2\times4+ & 5\times4^2 & \\
\hline
5^3+ & 3\times5^2\times4+ & 3\times5\times4^2+ & 4^3=729\text{, result.}
\end{array}$$

NOTE.—The teacher should carefully explain each step in this multiplication.

This example illustrates the following principle:

PRINCIPLE: *The cube of the sum of two numbers is equal to the cube of the first plus three times the square of the first times the second plus three times the first times the square of the second plus the cube of the second; or,*

$$\mathbf{(1st+2d)^3=1st^3+3\times 1st^2\times 2d+3\times 1st\times 2d^2+2d^3.}$$
(Commit.)

7. $23^3=(\ )?$

Process: $23^3=(20+3)^3=20^3+3\times 20^2\times 3+3\times 20\times 3^2+3^3=$ $8000+3600+540+27=12167$, result.

NOTE.—These two principles have been here presented, not so much for their practical value in *involution*, as to prepare for their use in *evolution*—extracting square root and cube root.

EXERCISE LXXIX.

1. $11^2=(7+4)^2=(\ )?$
2. $10^2=(3+7)^2=(\ )?$
3. $13^2=(5+8)^2=(\ )?$
4. $21^2=(20+1)^2=(\ )?$
5. $34^2=(30+4)^2=(\ )?$
6. $8^3=(3+5)^3=(\ )?$
7. $12^3=(10+2)^3=(\ )?$
8. $14^3=(10+4)^3=(\ )?$
9. $23^3=(20+3)^3=(\ )?$
10. $46^3=(40+6)^3=(\ )?$

C. EVOLUTION.

104. Definition.—Evolution is the process of finding a required root of a given power.

The number itself is the *first root;* one of the two equal factors which compose a number is its *second* or *square root;* one of the three equal factors which compose a number is its *third* or *cube root;* one of the four equal factors which compose a number is its *fourth root;* and so on.

The required root of a given number may be indicated in either of two ways:

(1) By the use of the *radical sign,* $\sqrt{\ }$, with a small figure or figures placed between the two parts to indicate the degree of the root. Thus,

$\sqrt[2]{121}$, the square root of 121.
$\sqrt[3]{1728}$, the cube root of 1728.
$\sqrt[4]{256}$, the fourth root of 256.

(2) By a fractional exponent. Thus,

$121^{\frac{1}{2}}$, the square root of 121.
$1728^{\frac{1}{3}}$, the cube root of 1728.
$256^{\frac{1}{4}}$, the fourth root of 256.

NOTE.—The radical sign without the small *2* is commonly used to indicate the *square* root of a number. Thus,

$\sqrt{64}$, the square root of 64.

1. SQUARE ROOT.

105. First Process.—The square roots of integral squares can usually be obtained by factoring. When a number is separated into two equal factors, one of the factors is the square root of the number.

EXAMPLES.

1. $\sqrt{225}, = (\ \ )$?

Process: $225 = 5\times5\times3\times3 = 15\times15$.
$\therefore \sqrt{225} = 15$, result.

2. $\sqrt{144\times64} = (\ \ )$?

Process: $144\times64 = 12\times12\times8\times8 = 96\times96$.
$\therefore \sqrt{144\times64} = 96$, result.

EXERCISE LXXX.

1. Commit to memory the square root of 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Find by factoring the square root of—

2. 900	*5.* 324	*8.* 81×25
3. 625	*6.* 576	*9.* 676×100
4. 256	*7.* 1296	*10.* 144×729

106. Second Process.—When numbers are large, or when they are not perfect squares, the process to be employed

in extracting the square root is developed from the following equation, which was learned in Involution:

$$(1st+2d)^2=1st^2+2\times 1st\times 2d+2d^2.$$

Preparatory to the development of this process, the following principles should be learned:

PRINCIPLES: *1. When an integral number is separated, from right to left, into periods of two figures each, the number of periods thus formed will equal the number of integral places in the square root of the number.*

NOTE.—This principle holds, even though there may be but one figure in the left-hand period.

This principle may be illustrated as follows:

$$\sqrt{1}=1,$$
$$\sqrt{1'00}=10,$$
$$\sqrt{1'00'00}=100,$$
$$\sqrt{1'00'00'00}=1000.$$

Thus, it appears that for integers less than 1′00, the square root cannot have more than one integral place; for integers less than 1′00′00, the square root cannot have more than two integral places; and so on.

2. When an integer is separated from right to left into periods of two figures each, the square root of the largest integral square in the left period will give the first or left figure of the square root of the number; the square root of the largest integral square of the number formed by the first two periods will give the first two figures of the square root of the number; and so on.

This principle may be illustrated as follows:

$$236^2=55696;\ \text{then,}\ \sqrt{5'56'96}=236.$$

The largest integral square in 5 is 4.

$$\sqrt{4}=2.$$

Then, 2 is the first figure of the square root, 236.

The largest integral square in 556 is 529.

$$\sqrt{529} = 23.$$

Then, 3 is the next figure of the square root, 236.

EXAMPLE.

Find the square root of 288369.

STEP 1.—*To find the first or left figure of the root.*

Process :

$$\begin{array}{l|l} 28'83'69 & 5 \\ \underline{25} & \\ 3 & \end{array}$$

Explanation : By separating the number into periods of two figures each, it appears that the square root will contain three figures (Prin. 1). The largest integral square in 28 is 25, of which 5 is the square root. Therefore, the first or hundreds figure of the root is *5* (Prin. 2).

STEP 2.—*To find the second figure of the root.* By Step 1, it was found that the first figure of the root is *5*; then, the number represented by the first two figures of the root must lie between 50 and 60, and the square of this number is the largest integral square in 2883, the first two periods of the number whose root is to be found (Prin. 2). If this number, which lies between 50 and 60, be represented as the sum of two numbers, then its square may be represented by the following equation:

$$(1st+2d)^2 = 1st^2 + 2 \times 1st \times 2d + 2d^2; \text{ or,}$$
$$(1st+2d)^2 = 1st^2 + 2d(2 \times 1st + 2d)$$

Now, since (1st+2d) represents a number lying between 50 and 60, let

$$1st = 50,$$

and proceed to find 2d.

Process :

		28′83′69	50+3=53.
$1st^2 = 2500$ (to be subtracted).........		25 00	
$2 \times 1st = 100$ (trial divisor)............	100	383	
$2d = 3$ (found by trial)...............	3		
$2 \times 1st + 2d = 103$ (complete divisor)...	103		
$2d(2 \times 1st + 2d) = 309$ (to be subtracted)		309	
		74	

Explanation: After subtracting the 2500 from the first two periods, there is left 383. Now, 383 must contain $100 \times 2d + 2d^2$. Since the *2d* represents the units of the second place, $2d^2$ is small as compared to $100 \times 2d$; finding how many times 383 contains 100 will practically find the *2d*. Using 100 for a *trial divisor*, proceed just as in finding the quotient figure in long division. The *2d* is found to be *3*. The *complete divisor* is formed by adding the *3* to the *100*, making 103. This complete divisor multiplied by the *3* in the root gives 309, the number to be subtracted.

STEP 3.—*To find the third or units figure of the root.* By Steps 1 and 2, the first and second figures of the root are 5 and 3 respectively. Then, the number represented by the three figures of the root must lie between 530 and 540, and the square of this number must not be greater than 288369. As in Step 2, consider this root as made up of two parts, and its square as represented by the equation—

$$(1st+2d)^2 = 1st^2 + 2d(2 \times 1st + 2d).$$

Let $1st = 530$,

and proceed to find the *2d* as before.

Process:		28′83′69	$530+7=537$, result.
$1st^2 = 280900$		28 09 00	
$2 \times 1st = 1060$	1060	74 69	
$2d = 7$	7		
$2 \times 1st + 2d = 1067$	1067		
$2d(2 \times 1st + 2d) = 7469$		7469	

EXERCISE LXXXI.

By steps as above, find:

1. $\sqrt{1849}$ *3.* $\sqrt{478864}$

2. $\sqrt{19044}$ *4.* $\sqrt{950625}$

107. Second Process Shortened.—The process of extracting square root, as just given, may be much shortened

(1) by consolidating the several steps into one continuous process. In example, p. 186, 25 subtracted in Step 1 amounts to the same as the 2500 subtracted in Step 2, and the 280900 subtracted in Step 3 amounts to the same as the 2500 and the 309 subtracted in Step 2; (2) by dropping all 0's that do not affect results.

EXAMPLES.

1. $\sqrt{288369}=(\ \ )$?

Process:

```
     28′83′69|537
     25
 103 |383
     |309
1067 |7469
     |7469
```

Explanation: (1) Separate the number as before, into periods of two figures each.

(2) $5\times5=25$. Place *5* in the root, subtract 25 from 28, and bring down the next period.

(3) $2\times5=10$. Place 10 on the left for a *trial* divisor, and try 10 into 38 (instead of 100 into 383 as before). $38\div10=3+$. Then, 3 is the second figure of the root. Place the *3* on the right of the 5 in the root and on the right of the *10* in the divisor. Multiply 103 by 3, write the product 309 beneath the 383, subtract, and bring down the next period.

(4) $2\times53=106$. Place the 106 on the left for a trial divisor, and proceed as before.

NOTE.—The teacher should assist in comparing this *shortened process* with the *process by steps*, and explain all differences not clear to the pupil.

2. $\sqrt{611.5729}=(\ \ )$?

Process:

```
     6′11.57′29|24.73
     4
  44 |211
     |176
 487 |3557
     |3409
4943 |14829
     |14829
```

NOTE.—In decimals, the periods are formed from the decimal point to the right. Otherwise, the process is the same as in integers.

When a number is not an exact power, 0's may be added for decimal places, and the process continued at will.

3. $\sqrt{5793649} = (\ \ )$?

Process:

$$\begin{array}{r|l}
 & 5'79'36'49 \ \underline{|2407}, \text{ result.} \\
 & 4 \\
\hline
44 & 179 \\
 & \underline{176} \\
\cancel{48} & 33649 \\
4807 & \underline{33649}
\end{array}$$

Explanation: If, after bringing down a period, the number thus formed is too small to contain the trial divisor, place a *0* in the root, bring down the next period, cancel the old trial divisor, form a new trial divisor, and proceed as before.

4. $\sqrt{\frac{625}{1024}} = (\ \ )$?

Process: (1) $\sqrt{625} = \sqrt{25 \times 25} = 25$

(2) $\sqrt{1024} = \sqrt{32 \times 32} = 32$

$\therefore \sqrt{\frac{625}{1024}} = \frac{25}{32}$, result.

Explanation: The square root of a common fraction is the square root of its numerator divided by the square root of its denominator. *This follows as a reverse of the process of squaring a common fraction.* Common fractions whose terms are not integral squares, should be reduced to decimals, and their square roots should be extracted as in No. 5, below.

5. $\sqrt{.043} = (\ \ )$? (Result true to 4 decimal places.)

$$\begin{array}{r|l}
 & .04'30'00'00 \ \underline{|.2073+}, \text{ result.} \\
 & 4 \\
\hline
\cancel{4} & 3000 \\
407 & \underline{2849} \\
4143 & 15100 \\
 & \underline{12429}
\end{array}$$

Explanation: (1) When the right-hand period of a decimal is not complete, it should be made so by annexing 0's.

(2) When there is a remainder after using the last period in either decimals or integers, periods of 0's may be annexed and the process continued at will.

EXERCISE LXXXII.

Find the value of—

1. $\sqrt{6889}$

2. $\sqrt{5625}$

3. $\sqrt{3136}$

4. $\sqrt{3025}$

5. $\sqrt{62001}$

6. $\sqrt{93636}$

7. $\sqrt{4556.25}$

8. $\sqrt{16.096144}$

9. $\sqrt{.1406850064}$

10. $(\frac{361}{1764})^{\frac{1}{2}}$

11. $(\frac{729}{1156})^{\frac{1}{2}}$

12. $(\frac{18225}{444889})^{\frac{1}{2}}$

Obtain results true to three decimal places :

13. $2^{\frac{1}{2}}$	*17.* $\sqrt{3.141592}$	*21.* $(\frac{2}{3})^{\frac{1}{2}}$
14. $3^{\frac{1}{2}}$	*18.* $\sqrt{45.7}$	*22.* $(42\frac{1}{7})^{\frac{1}{2}}$
15. $5^{\frac{1}{2}}$	*19.* $\sqrt{326.004}$	*23.* $(231\frac{11}{27})^{\frac{1}{2}}$
16. $11^{\frac{1}{2}}$	*20.* $\sqrt{.0007}$	*24.* $(33.47\frac{1}{2})^{\frac{1}{2}}$

2. CUBE ROOT.

108. First Process.—The cube roots of integral cubes can usually be obtained by factoring. When a number is separated into three equal factors, one of the factors is the cube root of the number.

EXAMPLES.

1. $\sqrt[3]{1728}=(\quad)$?

Process : $1728=4\times4\times4\times3\times3\times3=12\times12\times12$.

$\therefore \sqrt[3]{1728}=12$, result.

2. $\sqrt[3]{64\times512}=(\quad)$?

Process : $64\times512=4\times4\times4\times4\times4\times4\times2\times2\times2=32\times32\times32$.

$\therefore \sqrt[3]{64\times512}=32$, result.

EXERCISE LXXXIII.

1. Commit to memory the cubes of the integers from 1 to 10.

By factoring, find —

2. $\sqrt[3]{2744}$.	*5.* $\sqrt[3]{3375}$.	*8.* $\sqrt[3]{27\times729}$.
3. $\sqrt[3]{1331}$.	*6.* $\sqrt[3]{4096}$.	*9.* $\sqrt[3]{216\times1000}$.
4. $\sqrt[3]{4913}$.	*7.* $\sqrt[3]{8\times729}$.	*10.* $\sqrt[3]{512\times343}$.

109. Second Process.—When numbers are large or when they are not perfect cubes, the process to be employed in extracting the cube root is developed from the following equation, which was learned in Involution:

$$(1st+2d)^3=1st^3+3\times1st^2\times2d+3\times1st\times2d^2+2d^3.$$

Preparatory to the development of the process, the following principles should be learned:

PRINCIPLES: *1. When an integral number is separated from right to left into periods of three figures each, the number of periods thus formed will equal the number of integral places in the cube root of the number.*

NOTE.—This principle holds even though there may be but one or two figures in the left-hand period.

To illustrate:

$$\sqrt[3]{1}=1.$$
$$\sqrt[3]{1'000}=10.$$
$$\sqrt[3]{1'000'000}=100.$$
$$\sqrt[3]{1'000'000'000}=1000.$$

Thus, it appears that for integers less than 1'000, the cube root cannot have more than one integral place; for integers less than 1'000'000 the cube root cannot have more than two integral places; and so on.

2. When an integer is separated from right to left into periods of three figures each, the cube root of the largest integral cube in the left period will give the first or left figure of the cube root of the number; the cube root of the largest integral cube of the number formed by the first two periods will give the first two figures of the cube root of the number; and so on.

To illustrate:

$$236^3=13'144'256; \text{ then, } \sqrt[3]{13'144'256}=236.$$

The largest integral cube in 13 is 8.

$$\sqrt[3]{8}=2.$$

Then, 2 is the first figure of the cube root, 236.

The largest integral cube in 13144 is 12167.

$$\sqrt[3]{12167}=23.$$

Then, 3 is the second figure of the root, 236.

EXAMPLE.

Find the cube root of 154854153.

STEP 1.—*To find the first or left figure of the root.*

Process:
$$\begin{array}{l} 154'854'153(5 \\ \underline{125} \\ \ \ 29 \end{array}$$

Explanation: By separating the number into periods of three figures each, it appears that the cube root will contain three figures (Prin. 1.) The largest integral cube in 154 is 125, of which 5 is the cube root. Therefore, the first or hundreds figure of the root is *5* (Prin. 2).

STEP 2.—*To find the second figure of the root.* By step 1 it was found that the first figure of the root is 5; then the number represented by the first two figures of the root must lie between 50 and 60, and the cube of this number is the largest integral cube in 154854 (Prin. 2).

If this number which lies between 50 and 60 be represented as the sum of two numbers, then its cube may be represented by the following equation:

$$(1st+2d)^3=1st^3+3\times 1st^2\times 2d+3\times 1st\times 2d^2+2d^3;\ \text{or,}$$
$$(1st+2d)^3=1st^3+2d(3\times 1st^2+3\times 1st\times 2d+2d^2).$$

Now, since (1st+2d) represents a number lying between 50 and 60, let

$$1st=50,$$

and proceed to find *2d*.

Process:		
	154′854′153	50+3=53.
$1st^3=125000$ (to be subtracted)	125000	
$3\times 1st^2=7500$ (Trial divisor)	7500	29854
$3\times 1st\times 2d=450$	450	
$2d^2=9$	9	
$3\times 1st^2+3\times 1st\times 2d+2d^2$ (Complete divisor)	7959	
$2d(3\times 1st^2+3\times 1st\times 2d+2d^2)$		23877
		5977

Explanation: After subtracting the 125,000 from the first two periods, there is left 29854. Now, 29854 must contain $2d(3\times 1st^2+3\times 1st\times 2d+2d^2)$. Of this expression, the *2d* represents the figure of the root sought, and $(3\times 1st^2+3\times 1st\times 2d+2d^2)$ represents the *complete divisor*. Since the *2d* represents

the units of the second place in the root, the second and third terms of the complete divisor are small as compared with the first. Therefore, $3 \times 1st^2$, or 7500, is used as a *trial divisor*. Finding the root figure is done in the same way as finding the quotient figure in long division; but since the *trial divisor* is somewhat smaller than the *complete divisor*, allowance must be made for this in selecting the figure of the root. The second figure of the root is found to be *3*. The *complete divisor* is found by adding to the 7500 the values of $3 \times 1st \times 2d$ and $2d^2$, or 450 and 9, making in all 7959. This complete divisor, multiplied by the root figure *3*, gives the number to be subtracted, 23877.

STEP 3.—*To find the third figure of the root.* The first and second figures of the root have already been found to be *5* and *3* respectively. Then, the number represented by the three figures for the root must lie between 530 and 540, and the cube of this number must not be greater than 154854153. As in Step 2, consider this root as made up of two parts and its cube as represented by the equation,

$$(1st+2d)^3 = 1st^3 + 2d(3 \times 1st^2 + 3 \times 1st \times 2d + 2d^2).$$

$$\text{Let } 1st = 530,$$

and proceed to find the *2d* as before.

		154′854′153	530 + 7 = 537, result.
$1st^3 = 148877000$		148877000	
$3 \times 1st^2 = 842700$	842700	5977153	
$3 \times 1st \times 2d = 11130$..................	11130		
$2d^2 = 49$..................	49		
$3 \times 1st^2 + 3 \times 1st \times 2d + 2d^2$	853879		
$2d(3 \times 1st^2 + 3 \times 1st \times 2d + 2d^2)$..........		5977153	

EXERCISE LXXXIV.

By steps as above, find—

1. $\sqrt[3]{274625}$.

2. $\sqrt[3]{7189057}$.

3. $\sqrt[3]{94818816}$.

4. $\sqrt[3]{2087336952}$.

110. Second Process Shortened.—The process of extracting cube root, as just given, may be much shortened (1) by consolidating the several steps into one continuous process and (2) by dropping all 0's that do not affect the result.

EXAMPLES.

1. $\sqrt[3]{154854153}=(\quad)$?

Process:

	154′854′153 | 537, result.
	125
7500	29854
450	
9	
7959	23877
842700	5977153
11130	
49	
853879	5977153

Explanation: (1) Separate the number as before into periods of three figures each.

(2) $5^3=125$. Place the *5* in the root, subtract the 125 from 154, and bring down the next period.

(3) *The trial divisor is formed by annexing a 0 to the root already found, squaring the result and multiplying the square by 3.* $50^2\times3=7500$. $29854\div7500=3+$. Place the *3* in the root. *The complete divisor is obtained by adding together the trial divisor, three times the last figure of the root times the remainder of the root with a 0 annexed, and the square of the last figure of the root.* $3\times3\times50=450$; $3^2=9$. $7500+450+9=7959$, complete divisor. Multiply the complete divisor by the last figure of the root, subtract the product, and bring down the next period.

(4) Proceed as in (3).

2. $\sqrt[3]{8489664}=(\quad)$?

Process:

	8′489′664 | 204
	8
~~1200~~	489664
120000	
2400	
16	
122416	489664

Explanation: If after bringing down a period, the number thus formed is too small to contain the trial divisor, place a *0* in the root, bring down the next period, cancel the old trial divisor, form a new trial divisor, and proceed as before.

3. $\sqrt[3]{\frac{512}{729}}=(\quad)$?

Process: $\sqrt[3]{\frac{512}{729}}=\frac{8}{9}$, result.

Explanation: The cube root of a common fraction is equal to the cube root of the numerator divided by the cube root of the denominator. Common fractions, whose terms are not integral cubes, should be reduced to decimals, and their cube roots should be extracted as in No. 4 or 5 below.

4. $\sqrt[3]{.074088} = (\quad)$?

Process:

$$
\begin{array}{r|l}
 & .074'088 \;\underline{|\, .42}, \text{ result.} \\
 & 64 \\ \hline
4800 & 10088 \\
240 & \\
4 & \\ \hline
5044 & 10088 \\ \hline
\end{array}
$$

Explanation: Separate the number *from the decimal point to the right* into periods of three figures each; then, proceed as in integers.

5. $\sqrt[3]{12.3458} = (\quad)$? (Result true to three decimal places.)

Process:

$$
\begin{array}{r|l}
 & 12'.345'800'000 \;\underline{|\, 2.311+}, \text{result.} \\
 & 8 \\ \hline
1200 & 4345 \\
180 & \\
9 & \\ \hline
1389 & 4167 \\ \hline
158700 & 178800 \\
690 & \\
1 & \\ \hline
159391 & 159391 \\ \hline
16008300 & 19409000 \\
6930 & \\
1 & \\ \hline
16015231 & 16015231 \\ \hline
 & 3393769
\end{array}
$$

Explanation: (1) When the right-hand period of a decimal is not complete, it should be made so by annexing 0's. (2) When there is a remainder after using the last period in either decimals or integers, periods of 0's may be annexed and the process continued at will.

EXERCISE LXXXV.

Find the value of:

1. $\sqrt[3]{15625}$

2. $\sqrt[3]{13824}$

3. $\sqrt[3]{1225043}$

4. $\sqrt[3]{517781627}$

5. $\sqrt[3]{189119224}$

6. $\sqrt[3]{1967221277}$

7. $\sqrt[3]{65304767683}$

8. $\sqrt[3]{997002999}$

9. $\sqrt[3]{260917119}$

10. $\sqrt[3]{3422470843}$

11. $\sqrt[3]{351895816000}$

12. $\sqrt[3]{35773897375}$

13. $\sqrt[3]{1881365963625}$

14. $\sqrt[3]{66335045871501}$

15. $\sqrt[3]{1225.043}$

16. $\sqrt[3]{529.475129}$

17. $\sqrt[3]{.013312053}$

18. $\left(\frac{8000}{91125}\right)^{\frac{1}{3}}$

19. $\left(\frac{512}{19683}\right)^{\frac{1}{3}}$

20. $\left(\frac{1728}{12167}\right)^{\frac{1}{3}}$

Obtain results true to three decimal places:

21. $2^{\frac{1}{3}}$

22. $4^{\frac{1}{3}}$

23. $3.6^{\frac{1}{3}}$

24. $.244^{\frac{1}{3}}$

25. $.006^{\frac{1}{3}}$

26. $(247\frac{2}{3})^{\frac{1}{3}}$

3. ROOTS OF HIGHER DEGREES.

111. Process.—If the index of a root has no other factors than 2's or 3's, that root may be extracted by applying processes already explained.

PRINCIPLE: *Any root may be extracted by extracting in succession the roots indicated by the factors of the index of that root.*

Explanation: The 4th root is the square root of the square root; the 6th root is the square root of the cube root; the 8th root is the square root of the square root of the square root; and so on.

EXAMPLES.

1. $\sqrt[6]{64} = (\ \ )$?

Process: (1) $\sqrt[3]{64} = 4$.

(2) $\sqrt{4} = 2$, result.

NOTE.—The order is not material. The square root may be extracted first, and then the cube root.

2. $\sqrt[12]{531441} = (\ \ )$?

Process: (1) $\sqrt[3]{531441} = 81$.

(2) $\sqrt{81} = 9$.

(3) $\sqrt{9} = 3$, result.

Roots may often be found by factoring and inspecting the result.

3. $\sqrt[5]{3125} = (\quad)$?

Process: $3125 = 5 \times 5 \times 5 \times 5 \times 5.$

$\therefore \sqrt[5]{3125} = 5$, result.

4. $\sqrt[4]{1401} = (\quad)$?

Process: $1401 = 7 \times 7 \times 7 \times 7.$

$\therefore \sqrt[4]{1401} = 7$, result.

EXERCISE LXXXVI.

1. $\sqrt[8]{256}$	*3.* $\sqrt[9]{40353607}$	*5.* $\sqrt[7]{78125}$
2. $\sqrt[4]{14641}$	*4.* $\sqrt[6]{531441}$	*6.* $\sqrt[5]{248832}$

D. NUMBERS EXPRESSED BY WORDS OR LETTERS.

112. Explanations.—In writing out the solutions of problems, a great many *words* are used to express numbers. Some of the words commonly used in this way are the following:

number	area
value	volume
price	work
cost	force
gain	time
loss	interest
weight	principal
length	rate
width	sum
thickness	amount, etc.

NOTE.—Suppose this problem is given: "Five times the cost of my horse is \$320; find its cost." The relation given in the problem may be properly expressed as follows:

$$5 \times \text{cost} = \$320.$$

Here the word "*cost*" refers to a particular number and "*5* × *cost*" is 5 times that number.

Not only are words used to express numbers, but *letters* are also thus used.

NOTE.— Suppose that

Cost of 1 apple = 1¢; then
Cost of 2 apples = 2¢;
Cost of 3 apples = 3¢; and in general,
Cost of any number of apples = that number of cents.

Now, if n be used for the word number, the expression becomes—

Cost of n apples = n¢.

In the same way it is common to use v for value, p for price, c for cost, etc.

It is as necessary to add, subtract, multiply, and divide numbers thus expressed as it is to perform these processes with numbers expressed in figures.

113. Addition.

EXAMPLES.

1.	*2.*	*3.*	*4.*	*5.*	*6.*
40	4×10	4 *tens*	$4 \times cost$	$4 \times c$	4 c
50	5×10	5 *tens*	$5 \times cost$	$5 \times c$	5 c
60	6×10	6 *tens*	$6 \times cost$	$6 \times c$	6 c
150, sum.	15×10, sum.	15 *tens*, sum.	$15 \times cost$, sum.	$15 \times c$, sum.	15 c, sum.

NOTE.—Carefully compare the above examples (1) as to form of expression and (2) as to results. Notice that "$4c$" is read *four c*, and means *4 c's* or $4 \times c$. As a number, $4c$ is the same as $4 \times c$.

7. Add $10 \times cost$, $7 \times cost$, $5 \times number$, $9 \times cost$, and $6 \times number$.

Process:
$$\begin{array}{l} 10 \times cost + 5 \times number \\ 7 \times cost + 6 \times number \\ 9 \times cost \\ \hline 26 \times cost + 11 \times number, \text{ result.} \end{array}$$

NOTE.—In this example there are five numbers to be added, three of one kind and two of another. Those of the same kind are united. The

resulting numbers cannot be united into one number; but the sign, +, placed between them, shows that they are to be considered together as one sum.

EXERCISE LXXXVII.

Add:

1.	*2.*	*3.*	*4.*
$5 \times weight$	$17 \times area$	$3 \times time$	$16 \times loss$
$7 \times weight$	$21 \times area$	$12 \times time$	$loss$
$9 \times weight$	$area$	$21 \times time$	$12 \times loss$
$15 \times weight$	$11 \times area$	$time$	$27 \times loss$

5.	*6.*	*7.*	*8*	*9.*
$13 \times g$	$75 \times c$	$12 \times amt.$	$53 \times f$	$18 \times th.$
$7 \times g$	$18 \times c$	$95 \times amt.$	$121 \times f$	$72 \times th.$
$25 \times g$	$3 \times c$	$37 \times amt.$	$137 \times f$	$128 \times th.$
$18 \times g$	$54 \times c$	$19 \times amt.$	$86 \times f$	$943 \times th.$
$20 \times g$	$75 \times c$	$95 \times amt.$	$243 \times f$	$th.$

10.	*11.*	*12.*	*13.*	*14.*
$72\,r$	$85\,l$	$18\,w$	$7\,v$	$24\,s$
$8\,r$	$37\,l$	$63\,w$	$89\,v$	$17\,s$
$143\,r$	$128\,l$	$172\,w$	$27\,v$	$12\,s$
$59\,r$	$374\,l$	$94\,w$	$34\,v$	$9\,s$

15. $12 \times work$, $17 \times work$, $9 \times work$, $7 \times sum$, $18 \times sum$, $14 \times work$, $26 \times sum$, $6 \times work$, $34 \times sum$, and $12 \times sum$.

16. $5 \times c$, $7 \times f$, $8 \times c$, $12 \times c$, $17 \times f$, $18 \times f$, and $9 \times c$.

17. $24\,w$, $72\,w$, $120\,w$, $74\,t$, $95\,t$, $71\,w$, and $39\,t$.

114. Subtraction.

EXAMPLES.

1.	*2.*	*3.*	*4.*
$25\times value$	$25\times v$	$25\,v$	$25\,p$
$17\times value$	$17\times v$	$17\,v$	$17\,p$
$8\times value$, dif.	$8\times v$, dif.	$8\,v$, dif.	$8\,p$, dif.

5. From $25\,c$ take $17\,n$.

Process:

$$\begin{array}{r} 25\,c \\ \underline{17\,n} \\ 25\,c-17\,n, \text{ result.} \end{array}$$

NOTE.—The minuend and subtrahend are not similar numbers; therefore, the difference can only be indicated by placing the sign, −, between them.

EXERCISE LXXXVIII.

Subtract:

1.	*2.*	*3.*	*4.*
$36\times weight$	$125\times time$	$146\times cost$	$105\times loss$
$12\times weight$	$57\times time$	$cost$	$49\times loss$

5.	*6.*	*7.*	*8.*	*9.*
$120\times g$	$224\times f$	$175\times amt.$	$365\times v$	$459\times w$
$38\times g$	$189\times f$	$48\times amt.$	$175\times v$	$299\times w$

10.	*11.*	*12.*	*13.*	*14.*
$317\,p$	$41\,t$	$111\,r$	$215\,s$	$512\,l$
$195\,p$	$28\,t$	$55\,r$	$77\,s$	$71\,l$

15.	*16.*	*17.*	*18.*	*19.*
$75\times cost$	$157\times No.$	$17\times w$	$139\,v$	$75\,l$
$28\times area$	$139\times price$	$14\times t$	$74\,p$	$38\,w$

115. Multiplication.

EXAMPLES.

1. $\begin{array}{l} 12 \textit{ tens} \\ \underline{\quad 6 \quad} \\ 72 \textit{ tens} \end{array}$ *Explanation:* 6×12 tens are 72 tens.

2. $\begin{array}{l} \textit{ten} \\ \underline{\quad 6} \\ 6 \times \textit{ten}, \text{ or } 6 \textit{ tens}, \text{ product.} \end{array}$

3. $\begin{array}{l} 12 \times \textit{cost} \\ \underline{\quad 6 \qquad\quad} \\ 72 \times \textit{cost}, \text{ product.} \end{array}$

4. $\begin{array}{l} 12 \times w \\ \underline{\quad 6 \qquad} \\ 72 \times w, \text{ product.} \end{array}$

5. $\begin{array}{l} 12\,n \\ \underline{\quad 6 \quad} \\ 72\,n, \text{ product.} \end{array}$

6. Multiply p by n.

Explanation: (1) $1 \times p = p$.
(2) $2 \times p = 2\,p$.
(3) $3 \times p = 3\,p$; and so on.
(4) $n \times p = np$, result.

Thus, writing letters together, as np, *indicates* the *product of n and p.* It is common also to use the expression $n \times p$ to indicate the product. Both are correct.

7. Multiply $5 \times p$ by n.

Result: $n \times 5 \times p = 5 \times n \times p$, or $5\,pn$.

NOTE.—If p and n are abstract, by Principle 4, p. 21, they may stand in any order; it is customary to put the numerical factors before the letters or words.

8. Multiply $7c$ by $9\,a$.

Result: $9\,a \times 7\,c = 9 \times a \times 7 \times c = 63\,ac$.

NOTE.—Since factors may stand in any order, the *7* and *9* may be put together, making *63*.

EXERCISE LXXXIX.

Multiply:

1.	*2.*	*3.*	*4.*
$7 \times cost$	$24 \times area$	$137\ tens$	$56 \times rate$
12	36	48	17

5.	*6.*	*7.*	*8.*	*9.*
$16 \times v$	$45 \times w$	$158 \times th$	$48 \times l$	$129 \times p$
5	18	11	15	29
10.	*11.*	*12.*	*13.*	*14.*
$5p$	$17p$	$15l$	$12pr$	$6lw$
n	r	w	t	h
15.	*16.*	*17.*	*18.*	*19.*
pt	lh	$5wh$	$6pt$	$7wl$
r	w	$4l$	$5 \times r$	$7 \times h$

116. Division.

EXAMPLES.

1. Divide 12 *tens* by 6.

Process: $6 \underline{|\ 12\ tens}$
$\quad 2\ tens$, result. or, $12\ tens \div 6 = \frac{12}{6}\ tens = 2\ tens$, result.

PRINCIPLE: *Dividing any factor of a continued multiplication divides the product.*

2. Divide the *cost* by 6.

$$Process:\ Cost \div 6 = \frac{Cost}{6}, \text{ result.}$$

NOTE.—The result can only be *indicated*, and the ***fractional form*** is usually employed.

3. Divide $5 \times c$ by 9.

Process : $5 \times c \div 9 = \frac{5 \times c}{9}$, or, $\frac{5c}{9}$, result.

4. Divide $16w$ by n.

Process : $16w \div n = \frac{16w}{n}$, result.

5. Divide $15l$ by $25w$.

Process : $15l \div 25w = \frac{15l}{25w} = \frac{3l}{5w}$, result.

NOTE.—Cancel any common factors found in both numerator and denominator.

6. $5ln \div 7wn = (\quad)$?

Process : $5ln \div 7wn = \frac{5ln}{7wn} = \frac{5l}{7w}$, result.

NOTE.—n is common to both numerator and denominator.

EXERCISE XC.

Perform the process indicated :

1. $64 \times cost \div 8$.
2. $96 \times price \div 12$.
3. $36 \times value \div 6$.
4. $20 \times weight \div 5$.
5. $7 \times cost \div 14$.
6. $12 \times price \div 72$.
7. $cost \div 8$.
8. $12p \div 30$.
9. $16 \times w \div w$.
10. $w \div l$.
11. $w \times l \div h$.
12. $5 \times i \div 10p$.
13. $i \div pr$.
14. $5 \times t \div 75$.
15. $12lw \div 16lh$.
16. $35i \div 7pt$.
17. $ipt \div prt$.
18. $12cn \div 27wn$.
19. $18rt \div 40pt$.
20. $120prt \div 144pnt$.

117. Involution.

EXAMPLES.

1. Square p.

Process : $p \times p = p^2$, result.

NOTE.—Read "p^2" *p square.*

2. Square the *number*.

Process : $Number \times number = (number)^2$, result.

3. Square $5a$.

Process : $5a \times 5a = 25a^2$, result.

4. Square $a+b$.

Process :

$$\begin{array}{r} a+b \\ a+b \\ \hline ab+b^2 \\ a^2+ab \quad \\ \hline a^2+2ab+b^2 \end{array}, \text{ result.}$$

Questions : Do this process and result agree with the principle on page 181? Can you repeat that principle from memory?

5. Cube $7l$.

Process : $7l \times 7l \times 7l = 343l^3$, result.

NOTE.—Read "l^3" *l cube.*

6. Cube $\frac{5a}{w}$.

Process : $\frac{5a}{w} \times \frac{5a}{w} \times \frac{5a}{w} = \frac{125a^3}{w^3}$, result.

7. $\left[\frac{7c}{4n}\right]^5 = (\quad)$?

Process : $\frac{7c}{4n} \times \frac{7c}{4n} \times \frac{7c}{4n} \times \frac{7c}{4n} \times \frac{7c}{4n} = \frac{16807c^5}{1024n^5}$, result.

8. $(a+b)^3 = (\quad)$?

Process :

$$\begin{array}{r} a+b \\ a+b \\ \hline ab+b^2 \\ a^2+ab \quad \\ \hline a^2+2ab+b^2 \\ a+b \\ \hline a^2b+2ab^2+b^3 \\ a^3+2a^2b+ab^2 \quad \\ \hline a^3+3a^2b+3ab^2+b^3 \end{array}, \text{ result.}$$

Questions : Do this process and result agree with principle, page 182? Can you repeat that principle from memory?

EXERCISE XCI.

Give results orally:

1. Square (1) n; (2) $2n$; (3) pn; (4) $\frac{7n}{w}$; (5) $\frac{8np}{5wa}$.

2. Square (1) $a+h$; (2) $v+p$; (3) $n+r$; (4) $w+l$.

3. Cube (1) n; (2) $5n$; (3) nl; (4) $\frac{5w}{3a}$; (5) $\frac{2lw}{3pt}$.

4. Cube (1) $a+h$; (2) $v+p$; (3) $n+r$; (4) $w+l$.

Give written processes:

5. $\left[\frac{7}{w}\right]^3$

6. $\left[\frac{9a}{14b}\right]^3$

7. $\left[\frac{126l}{149a}\right]^2$

8. $(a^2)^2$

9. $\left[\frac{4rl}{9vt}\right]^5$

10. $\left[\frac{5ab}{7cw}\right]^4$

11. $\left[\frac{11np}{16bc}\right]^3$

12. $\left[\frac{2n}{4ar}\right]^6$

13. $(2+n)^2$

14. $(3a+4b)^2$

15. $(w+5)^3$

16. $(4r+3n)^3$

118. Evolution.

EXAMPLES.

1. $\sqrt{a^2}=(\ \ )$?

Process: $\sqrt{a^2}=\sqrt{a\times a}=a$, result.

Question: What is the square root of a number?

2. $\sqrt{9a^2b^2}=(\ \ )$?

Process: $\sqrt{3ab\times 3ab}=3ab$, result.

3. $\sqrt{\frac{36a^2}{25c^2}}=(\ \ )$?

Process: $\sqrt{\frac{36a^2}{25c^2}}=\sqrt{\frac{6a}{5c}\times\frac{6a}{5c}}=\frac{6a}{5c}$, result.

4. $\sqrt[3]{64a^3} = (\ \ )$?

Process: $\sqrt[3]{64a^3} = \sqrt[3]{4a \times 4a \times 4a} = 4a$, result.

Question: What is the cube root of a number?

5. $\sqrt[3]{\dfrac{27a^3r^3}{216l^3w^3}} = (\ \ )$?

Process: $\sqrt[3]{\dfrac{27a^3r^3}{216l^3w^3}} = \sqrt[3]{\dfrac{3ar}{6lw} \times \dfrac{3ar}{6lw} \times \dfrac{3ar}{6lw}} = \dfrac{3ar}{6lw}$, result.

6. $\sqrt[4]{81w^4l^4} = (\ \ )$?

Process: $\sqrt[4]{81w^4l^4} = \sqrt[4]{3wl \times 3wl \times 3wl \times 3wl} = 3wl$, result.

7. $\sqrt{a^2+2ab+b^2} = (\ \ )$?

Process: $\sqrt{a^2+2ab+b^2} = \sqrt{(a+b)(a+b)} = a+b$, result.

NOTE.—"$(a+b)(a+b)$" $= a+b$ multiplied by $a+b$.

8. $\sqrt[3]{a^3+3a^2b+3ab^2+b^3} = (\ \ )$?

Process: $\sqrt[3]{a^3+3a^2b+3ab^2+b^3} = \sqrt[3]{(a+b)(a+b)(a+b)} = a+b$, result.

EXERCISE XCII.

Give results orally:

1. Square root of (1) l^2; (2) $4n^2$; (3) $16l^2$; (4) $\dfrac{64w^2}{r^2}$; (5) $\dfrac{p^2l^2}{81w^2}$.

2. Square root of (1) $l^2+2lr+r^2$; (2) $n^2+2nc+c^2$.

3. Cube root of (1) c^3; (2) $27b^3$; (3) $125w^3h^3$; (4) $\dfrac{64p^3}{v^3w^3}$; (5) $\dfrac{t^3r^3}{216n^3}$.

4. Cube root of (1) $p^3+3p^2r+3pr^2+r^3$; (2) $l^3+3l^2w+3lw^2+w^3$.

Give written process :

5. $\left[\dfrac{125\,c^3}{d^3}\right]^{\frac{1}{3}}$

6. $\left[\dfrac{121\,t^2}{p^2r^2w^2}\right]^{\frac{1}{2}}$

7. $\left[\dfrac{h^3d^3}{l^3w^3}\right]^{\frac{1}{3}}$

8. $\left[\dfrac{49}{2500\,n^2}\right]^{\frac{1}{2}}$

9. $(p^2+2\,rp+r^2)^{\frac{1}{2}}$

10. $(w^2+4\,w+4)^{\frac{1}{2}}$

11. $(t^3+3\,t^2r+3\,tr^2+r^3)^{\frac{1}{3}}$

12. $(n^3+3\,n^2r+3\,nr^2+r^3)^{\frac{1}{3}}$

119. Equations Containing Letters.—Formulas, which are equations containing letters, are often used in arithmetic. For example,

$$(1)\quad C=np.$$

Now, there are three numbers in this equation. If any two of them are known, the other may be found. For

$$(1)=(2)\quad np=C.$$

$$(2)\div n=(3)\quad p=\frac{C}{n}.$$

$$(2)\div p=(4)\quad n=\frac{C}{p}.$$

If the values of n and p be substituted (put in place of the letters) in (1), the result will express the value of C; if the values of C and n be substituted in (3), the result will express the value of p; and if the values of C and p be substituted in (4), the result will express the value of n.

Note.—When an equation, containing letters, is so arranged that one letter stands alone in the left member, that letter is said to be expressed in terms of the others. Thus, in (1), C is expressed in terms of p and n; in (3), p is expressed in terms of C and n; and in (4), n is expressed in terms of C and p.

EXAMPLES.

1. $A=lw$. (1) Express the value of w in terms of A and l. (2) Express the value of l in terms of A and w.

Process : (1) $A=lw$.
(1) = (2) $lw=A$.
(2) ÷ l = (3) $w=\frac{A}{l}$, 1st result.
(2) ÷ w = (4) $l=\frac{A}{w}$, 2d result.

2. $F=\frac{mv}{t}$. Express the values of m, v, and t, each in terms of the other numbers.

Process : (1) $F=\frac{mv}{t}$.
(1) = (2) $\frac{mv}{t}=F$.
t × (2) = (3) $mv=Ft$.
(3) ÷ v = (4) $m=\frac{Ft}{v}$, 1st result.
(3) ÷ m = (5) $v=\frac{Ft}{m}$, 2d result.
(3) = (6) $Ft=mv$.
(6) ÷ F = (7) $t=\frac{mv}{F}$, 3d result.

3. $432=16(2t-1)$. Find the value of t.

Process : (1) $432=16(2t-1)$.
(1) = (2) $16(2t-1)=432$.
(2) ÷ 16 = (3) $2t-1=27$.
Transposing, we get (4) $2t=28$.
$\frac{1}{2}$ of (4) = (5) $t=14$, result.

4. $S=\frac{1}{2}gt^2$. Express the values of g and t, each in terms of the other numbers.

Process : (1) $S=\frac{1}{2}gt^2$.
(1) = (2) $\frac{1}{2}gt^2=S$.
(2) ÷ $\frac{1}{2}t^2$ = (3) $g=\frac{2S}{t^2}$, 1st result.
(2) ÷ $\frac{1}{2}g$ = (4) $t^2=\frac{2S}{g}$
$\sqrt{(4)}$ = (5) $t=\sqrt{\frac{2S}{g}}$, 2d result.

5. $A=p+prt$. Express the values of p, t, and r.

Process: (1) $A=p+prt=p(1+rt)$.

(1) = (2) $p(1+rt)=A$.

(2) ÷ $(1+rt)$ = (3) $p=\frac{A}{1+rt}$, 1st result.

(1) = (4) $p+prt=A$.

Transposing, we get (5) $prt=A-p$.

(5) ÷ pr = (6) $t=\frac{A-p}{pr}$, 2d result.

(5) ÷ pt = (7) $r=\frac{A-p}{pt}$, 3d result.

EXERCISE XCIII.

1. $P=\frac{pB}{100}$. Express the values of p and B.

2. $F=Ma$. Express the values of M and a.

3. $W=Fd$. Express the values of F and d.

4. $W=Mda$. Express the values of M, d, and a.

5. $25\times30=15M$. Find the value of M.

6. $369.84=\frac{1}{2}g(24-1)$. Find the value of g.

7. $1302.48=16.08t^2$. Find the value of t.

8. $A=\frac{W}{550t}$. Express the values of W and t.

9. $V=lwh$. Express the values of l, w, and h.

10. $H^2=B^2+P^2$. Express the values of P and B.

11. $C=\frac{E}{R}$. Express the values of E and R.

12. $A=\pi R^2$. Express the value of R.

NOTE.—The Greek letter π (*pi*) is used for the number 3.1416.

13. Find the value of F in No. 3, if $W=75$ and $d=15$.

14. $I=prt$. Find the value of p, if I, r and t are respectively 192, $\frac{8}{100}$, 4.

15. Find the value of B in No. 10, if H and P are respectively 50 and 30.

II. STUDY OF PROBLEMS.

A. DEVELOPMENT AND USE OF FORMULAS.

120. General Numbers and General Problems.—If I say, "*The fence is 40 rods long*," I use a *particular number, 40 rods*. If I say, "The fence is l rods long," I use a *general number, l rods*, which may prove to be 40 rods, or some other number of rods. This *general* form of expressing a number that may vary in its numerical value is of great service in mathematics.

Examine the following problems:

1. What will 5 books cost, at \$2 each ?
2. What will 25 lb. of meat cost, at 10¢ per lb.?
3. Find the value of 140 A. of land, at \$40 per A.
4. What will 12 horses sell for, at \$75 each ?

These are all different problems, yet they all belong to the same type.

TYPE: *What will a given number of articles* cost, at a given price each ?

By using *letters* to represent the numbers in this type, it may be expressed in the following form:

What will n things cost, at \$$p$ (or p¢) each ?

Such a type, which is a representative of a great class of problems, is often spoken of as a *General Problem*. It is of great service, in solving particular problems, to know to what type they belong.

EXAMPLES.

1. What is the area of a field 50 rd. long, 30 rd. wide ? Give the type.

TYPE: What is the area of a surface (or field) l rd. long, w rd. wide?

NOTE.—In giving types, it is well to use initial letters, as far as convenient, to represent the numbers; as, l for *length* and w for *width*.

2. Find the interest on \$400 for $2\frac{1}{2}$ years at 7% per annum? Give the type.

TYPE: Find the interest on \$$p$ for t yr. at r % per annum.

3. Find the length of a solid of 3600 cu. ft., 20 ft. wide, 6 ft. thick.

TYPE: Find the length of a solid of v cu. ft., w ft. wide, t ft. thick.

EXERCISE XCIV.

Give the types:

1. Find 20% of \$500.

2. How many books will bring \$40 at \$5 each?

3. \$75 is 15% of what number?

4. What must be the width of one acre (160 sq. rd.), if it is 40 rd. long?

5. What force is required to add 10 ft. per second to the velocity of 15 lb. of matter?

NOTE.—Use a (acceleration) for the number of feet, and M (mass) for the number of pounds.

6. 75 is what % of 150?

7. I sold 20 lb. of butter for \$2. Find the price per lb.

8. Find the width of a wall of 2340 cu. ft., 130 ft. long, $1\frac{1}{2}$ ft. thick.

121. Solving the General Problem or Type.

The same reasoning and plan are used in solving the *general problem*, or *type*, as are used in solving the *particular problem*.

EXAMPLE.

Find the cost of n things, at \$$p$ each.

Solution: (1) Cost of n things, at \$$p$ each = \$( )? (Question.)

(2) Cost of 1 thing, at \$1 each = \$1. (Basis.)

$p \times (2) = (3)$ Cost of 1 thing, at \$$p$ each = \$$p$.

$n \times (3) = (4)$ Cost of n things, at \$$p$ each = \$$np$, result.

If the *cost* be represented by C and (4) be expressed in its abstract form, it becomes —

$$(5)\quad C=np.$$

This equation is called a **Formula,** and it expresses an abstract numerical *relation* that is always true.

RELATION: *The cost of a number of things of the same kind is equal to the product of the number of things and the price of one of the things.*

This formula relates three numbers: the *cost*, the *number* of things, and the *price* of each. When any two of these numbers are given, the third may be found by aid of the formula.

EXAMPLES.

1. Find the cost of 20 bu. of apples, at 30¢ per bu.

Solution: (1) $C=np$.
(2) $C=20\times 30=600$.
$\therefore$ the cost is 600¢, or \$6.

2. How many bushels of apples at \$.30 each can I buy for \$18?

Solution: (1) $C=np$.
(2) $18=.3\,n$.
(2) = (3) $.3\,n=18$.
(3) ÷ .3 = (4) $n=\frac{18}{.3}=60$.
$\therefore$ the reqd. answer is 60 bu.

3. If 50 bu. of apples cost \$15, find the price per bushel.

Solution: (1) $C=np$.
(2) $15=50p$.
(2) = (3) $50p=15$.
$\frac{1}{50}$ of (3) = (4) $p=\frac{15}{50}=.3$.
$\therefore$ the price is \$.3, or 30¢.

NOTE.—In examples 2 and 3, the work would be shortened by one equation if the formula had been turned around in the first equation so that the required number would appear in the left member.

EXERCISE XCV.

1. If n bu. of apples cost $\$C$, find the price per bu. **Answer:** $p=\frac{C}{n}$.

2. If $\$C$ are paid for apples at $\$p$ per bu., find the number of bushels bought. **Answer:** $n=\frac{C}{p}$.

Use answer to No. 1 as a formula.

3. Find the price per acre, if 120 A. of land sell for \$1440.

4. I sold 5000 lb. of bacon for \$400. Find the price per lb.

Use answer to No. 2 as a formula:

5. I spend \$15.60 for turkeys, at \$.65 each. How many did I buy?

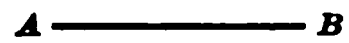
6. A dealer invests \$975 in buggies, at \$75 each. How many did he buy?

B. MENSURATION.

1. LINES AND ANGLES.

122. Definitions.—Define a line. (See p.149.)

A line whose length has but one direction is a **Straight Line.**



AB, a line; also, a straight line.

A line whose length changes direction at every point is a **Curved Line.**

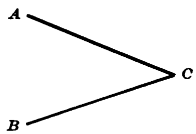
CD, a curved line.

Two lines that never approach each other are **Parallel.**

AB and *CD*, parallel lines.

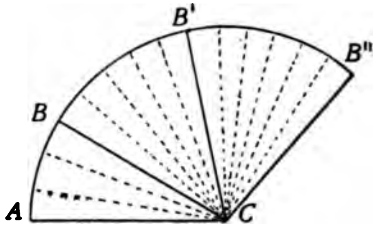
NOTE.—Lines that are parallel have the same direction and never meet. If two lines meet, they form one or more angles, and have *different directions.*

An **Angle** is the difference of direction between two lines that meet.

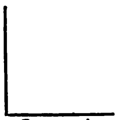


ACB, an angle.

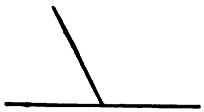
NOTE 1.—As line BC, starting from the position AB, swings around point C, the direction of line BC is continually changing, and the difference of direction between line AC and line BC is continually growing larger. Angle ACB is smaller than angle ACB'; angle ACB' is smaller than angle ACB''; and so on.



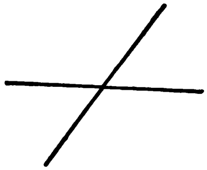
NOTE 2.—If two lines, coming together, stop at the point of meeting, they form but *one angle*. If only one line stops at the point of meeting, they form *two angles*. If neither line stops at the point of meeting, they form *four angles*.



One angle.

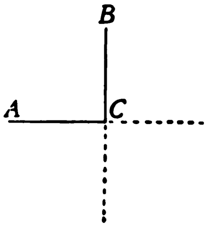


Two angles.

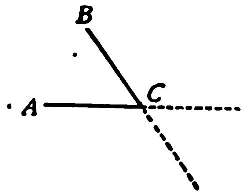


Four angles.

When the directions of two lines that meet are such that, if the lines are continued past the point of meeting, they form *equal angles*, the lines are **Perpendicular** to each other. When the angles thus formed are not all equal, the lines are **Oblique** to each other.

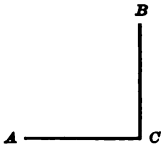


AC and BC are perpendicular to each other.

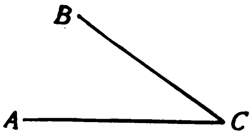


AC and BC are oblique to each other.

NOTE.—The lines forming an angle are called its *sides*, and the point of meeting of the sides is called the *vertex* of the angle.



ACB, a right angle.



ACB, an acute angle.

ACB, an obtuse angle.

A **Right Angle** has its sides perpendicular to each other.

An **Acute Angle** is smaller than a right angle.

An **Obtuse Angle** is larger than a right angle.

NOTE.—The common unit of angular measure is the *degree*, marked (°).

$$1 \text{ right angle} = 90^\circ$$

A line parallel with the regular lines of writing or print is a **Horizontal Line.**

A line perpendicular to the regular lines of writing or print is a **Vertical Line.**

AB, a horizontal line.

CD, a vertical line.

NOTE.—The word "horizontal" came to be applied to lines, probably because in writing upon upright surfaces such as blackboards, such lines are parallel with the horizon; but if the surface on which you write is parallel with the horizon, all lines made on that surface are also parallel with the horizon.

EXERCISE XCVI.

Define and draw (free-hand):

1. A line.
2. A straight line.
3. A curved line.
4. Two parallel lines.
5. An angle.
6. Two perpendicular lines.
7. Two oblique lines.
8. A right angle.
9. An acute angle.
10. An oblique angle.
11. A horizontal line.
12. A vertical line.

NOTE.—Become familiar with these lines and angles before passing to the next subject.

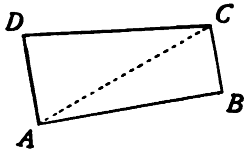
2. PLANE SURFACES.

123. Parallelogram.—Define *Surface*. (See p. 154.)

A **Plane** is a surface each of whose dimensions has but *one direction*.

A **Polygon** is a plane bounded by *straight lines*. The straight lines are called *sides*, and the points where the sides meet are called *angles* or *vertices*.

A **Quadrilateral** is a polygon of four sides. The lines joining the alternate angles are called *diagonals*.

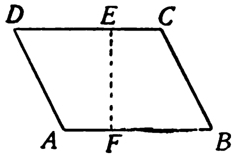


ABCD, a quadrilateral; *AC* a diagonal.

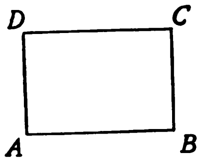
A **Parallelogram** is a quadrilateral whose opposite sides are parallel. The line on which a parallelogram rests is usually called its *base*, and the perpendicular distance from the base to the opposite side (or opposite side produced) is called its *altitude*.

A **Rectangle** is a parallelogram whose angles are right angles.

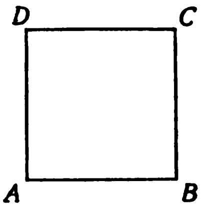
A **Square** is a rectangle whose sides are equal.



ABCD, a parallelogram and rhomboid; *AB*, the base; *EF*, the altitude.



ABCD, a rectangle; *AB*, the base; *DA* or *CB*, the altitude.

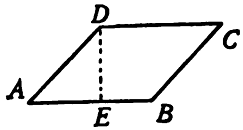


ABCD, a square; *AB*, base; *CB* or *AD*, the altitude.

A **Rhomboid** is a parallelogram whose angles are not right angles.

For form of rhomboid, see parallelogram.

A **Rhombus** is a rhomboid whose sides are equal.



ABCD, a rhombus; *AB*, the base; *DE*, the altitude.

EXERCISE XCVII.

Define and draw (free-hand, but carefully):

1. A quadrilateral.
2. A parallelogram.
3. A rectangle.
4. A square.
5. A rhomboid.
6. A rhombus.

Define:

7. A surface.
8. A plane.
9. A side.
10. A vertex.
11. A diagonal.
12. A polygon.
13. A base.
14. An altitude.

NOTE.—These terms must be defined as applied to polygons.

PROBLEM 1: *Letting* b *represent the length and* a *the width of a rectangle, develop the formula for the area.*

BY EQUATIONS.

Solution: (1) Area of a surface b ft. l., a ft. w. = ( ) sq. ft.? (Question).
(2) Area of a surface 1 ft. l., 1 ft. w. = 1 sq. ft. (Basis).
$b\times(2)=(3)$ Area of a surface b ft. l., 1 ft. w. $=b$ sq. ft.
$a\times(3)=(4)$ Area of a surface b ft. l., a ft. w. $=ab$ sq. ft., answer.

BY PROPORTION.

Solution: (1) Area of a surface b ft. l., a ft. w = ( ) sq. ft.? (Question.)
(2) Area of a surface 1 ft. l., 1 ft. w. = 1 sq. ft. (Basis.)
(3) $b\times a : 1\times 1 :: (\ ) : 1$?
(4) $\frac{b\times a\times 1}{1\times 1}=ba$, answer.

Using A for area, the result may be written as follows:

$$A=ba.$$

RELATION I. *Abstractly, the area of a rectangle is equal to the product of the base and altitude.*

EXAMPLES.

1. Find the area of a wall 20 ft. long and $7\frac{1}{2}$ ft. high.

Solution: (1) $A = ba$.
(2) $A = 20 \times 7\frac{1}{2} = 150$.
$\therefore$ the required area is 150 sq. ft.

2. A rectangular field is 80 rd. long and contains 10 A. How wide is it?

Solution: (1) 10 A = 1600 sq. rd.
(2) $ab = A$.
(3) $80a = 1600$.
(3) ÷ 80 = (4) $a = 20$.
$\therefore$ the required width is 20 rd.

3. A sidewalk 4 ft. wide contains 960 square feet. How many yards long is it?

Solution: (1) $ab = A$.
(2) $4b = 960$.
$\frac{1}{4}$ of (2) = (3) $b = 240$.
$\therefore$ the required length is 240 ft. or 80 yd.

NOTE.—Before applying the formula, the dimensions must be expressed in *linear units* of the same denomination; then, the area will be *square units* of that denomination. Use only the *corresponding abstract numbers* in the formula.

PROBLEM 2: *Develop the formula for the square.*

Development: A square is a rectangle, whose base (b) and altitude (a) are equal. If we call one side of the square s, we can substitute s for a and b in the formula, thus:

$$A = s \times s\text{; or,}$$
$$A = s^2.$$

RELATION II. *Abstractly, the area of a square equals the square of one of its sides.*

EXAMPLES.

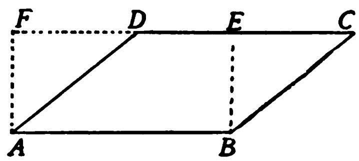
1. Find the area of a square one of whose sides is 46 ft.

Formula: (1) $A=s^2$.
(2) $A=46^2=2116$.
$\therefore$ the required area is 2116 sq. ft.

2. The area of a square is 102400 sq. rd. Find the length of one side.

Formula: (1) $s^2=A$.
(2) $s^2=102400$.
$\sqrt{(2)}=$ (3) $s=320$.
$\therefore$ the required result is 320 rd.

PROBLEM 3: Develop the formula for the parallelogram.



Development: Let $ABCD$ be a parallelogram, whose base is AB. Draw BE from B perpendicular to CD, cutting off the part BEC. Place this part on the left as shown in AFD, forming the rectangle $ABEF$. Thus, the parallelogram has been changed to a rectangle, whose *base* and *altitude* are the same as those of the parallelogram. Then, the area of the parallelogram is equal to that of the rectangle; or,

$$A=ba.$$

RELATION III. *The same as that of the formula for a rectangle.*

EXAMPLES.

1. Find the area of a parallelogram whose base and altitude are 25 ft. and 50 ft. respectively.

Formula: (1) $A=ba$.
(2) $A=25\times50=1250$.
$\therefore$ the reqd. area is 1250 sq. ft.

2. The area of a parallelogram is 1440 sq. rd. What is its length, if it is only 16 rd. wide?

Solution: (1) $16\,b=1440$. (Why?)
(2) $b=\frac{1440}{16}=90$.
$\therefore$ the reqd. length is 90 rd.

NOTE.—In rapid work, the formula may be omitted, as in example 2; but never until its use is thoroughly understood—not until the pupil knows just what *substituting in a formula* means.

EXERCISE XCVIII.

1. In what respect is a square a parallelogram? In what respect a rectangle?

2. What is the difference between a square and a rhombus?

3. Is it true that the formula $A=ab$, will apply to a rectangle? A square? A rhomboid? A rhombus? Why?

4. Show how the formula $A=ab$ is modified to $A=s^2$ for the square.

5. Carefully draw and cut out of paper a parallelogram; then, cut it into two parts and put the parts together to form a rectangle as is indicated in problem 3. Can every parallelogram (not already a rectangle) be thus reduced to a rectangle?

6. In figure in problem 3, does the perpendicular BE have to be erected at B, or may it be erected somewhere else along the base?

7. Commit the relations I to III.

Find the area of the following parallelograms:

8. Base 90 ft., altitude 37 ft.

9. Base 3 ft., altitude 12 in.

NOTE.—Dimensions must be expressed in the same denomination before applying the formula.

10. Base 5 mi., altitude 700 rd.

11. Base $5\frac{1}{2}$ yd., altitude $16\frac{1}{2}$ ft.

In each of the following parallelograms, find the dimension not expressed:

12. Area 2700 sq. m., altitude 60 m.

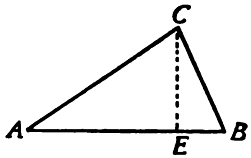
13. Area 960 A., base 600 rd.

14. Area 1156 sq. ft., base 34 ft.

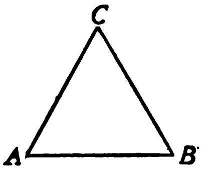
15. Area 100500 sq. yd., altitude 450 ft.

16. Area 57500 sq. cm., base 46 dm.

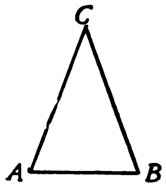
124. Triangles.—A **Triangle** is a polygon of three sides. Any side of a triangle may be considered its *base*, and the perpendicular from any vertex to the opposite base is the *altitude* corresponding to that base.



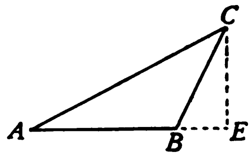
ABC, a triangle; *AB*, the base; *CE*, the altitude.



ABC, an equilateral triangle; also, equiangular.



ABC, an isosceles triangle.



ABC, a scalene triangle; *AB*, the base; *CE*, the altitude.

An **Equilateral Triangle** has its three sides equal.

An **Isosceles Triangle** has two of its sides equal.

A **Scalene Triangle** has no two of its sides equal.

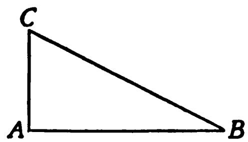
Every triangle has three angles, which together measure just 180°.

An **Equiangular Triangle** has its three angles equal.

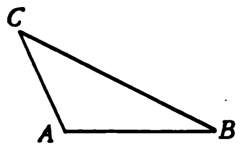
(For form of equiangular triangle, see equilateral triangle.)

NOTE.—Each angle of an equiangular triangle measures just 60°.

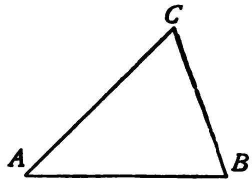
A **Right Triangle** has one of its angles a right angle. The side opposite the right angle is called the *hypotenuse;* the other two sides are usually called *base* and *perpendicular*.



ABC, a right triangle; *AB*, the base; *CA*, the perpendicular or altitude; *CB*, the hypotenuse.



ABC, an obtuse triangle.



ABC, an acute triangle.

An **Obtuse Triangle** has one of its angles an obtuse angle.

An **Acute Triangle** has all of its angles acute angles.

EXERCISE XCIX.

Define and draw :

1. A triangle.

2. An equilateral triangle.

3. An isosceles triangle.

4. A scalene triangle.

5. An equiangular triangle.

6. A right triangle.

7. An obtuse triangle.

8. An acute triangle.

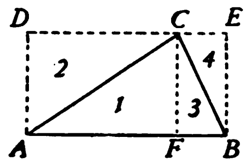
9. Define (1) base, (2) altitude, (3) vertex, (4) hypotenuse, (5) perpendicular.

10. How many degrees in all the angles of a triangle ?

11. Can two of the angles of a triangle be right angles ? Can two of them be obtuse ? Why ?

12. Construct an obtuse triangle and draw its three altitudes.

PROBLEM 1: *Develop the formula for the triangle.*



Development : Let ABC be a triangle. Upon the base AB construct the rectangle $ABED$, just as high as the triangle. From C drop a perpendicular CF to the base AB. Parts *1* and *2* are equal, and parts *3* and *4* are equal. Therefore, the triangle is just half as large as the rectangle. But the base and altitude of the triangle are the same as those of the rectangle. **Therefore,**

$$A=\frac{ab}{2}.$$

RELATION IV. *Abstractly, the area of a triangle is equal to the product of its base and altitude, divided by 2.*

EXAMPLES.

1. Find the area of a triangle whose base and altitude are respectively 40 ft. and 15 ft.

Solution: $A=\frac{40\times 15}{2}=300.$

$\therefore$ the required area is 300 sq. ft.

2. The base of a triangle is 75 ft., its area is 400 sq. yd. Find its altitude.

Solution: (1) 400 sq. yd. = 3600 sq. ft.

(2) $\frac{75\,a}{2}=3600.$

(3) $a=\frac{3600\times 2}{75}=96.$

$\therefore$ the required altitude is 96 ft.

RELATION V. *In right triangles, abstractly, the square of the hypotenuse equals the square of the base plus the square of the perpendicular.*

NOTE.—The development of this relation is too difficult to be given before the pupil has studied geometry. It gives us an important formula:

$$H^2=B^2+P^2.$$

EXAMPLES.

1. The base of a right triangle is 36 ft.; the altitude, 27 ft. Find the hypotenuse.

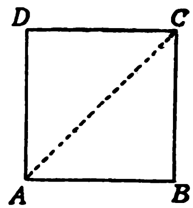
Solution: (1) $H^2=B^2+P^2.$

(2) $H^2=36^2+27^2=2025.$

(3) $H=\sqrt{2025}=45.$

$\therefore$ the required result is 45 ft.

2. Find the diagonal of a square, one of whose sides is 32 inches.



NOTE.—$ABCD$ is a square. AC, the diagonal, is the hypotenuse of the right triangle CAB. $AB=32$, $BC=32$.

Solution: (1) $H^2=32^2+32^2=2048$.
(2) $H=\sqrt{2048}=45.25+$.
$\therefore$ the diagonal is 45.25 in.

3. Find the base of a right triangle whose perpendicular is 52 rd. and whose hypotenuse is 65 rd.

Solution: (1) $B^2+P^2=H^2$.
(2) $B^2+52^2=65^2$.
(3) $B^2=65^2-52^2=1521$.
(4) $B=\sqrt{1521}=39$.
$\therefore$ the required result is 39 rd.

4. The diagonal of a square is 60 yd. Find one side.

NOTE.—The diagonal is the hypotenuse of the right triangle whose sides are equal.

Solution: (1) $s^2+s^2=H^2$; or,
(2) $2s^2=H^2=60^2=3600$.
$\frac{1}{2}$ of (2)=(3) $s^2=1800$.
(4) $s=\sqrt{1800}=42.426+$.
$\therefore$ the reqd. side is 42.426 yd.

EXERCISE C.

1. Study the formula for the area of a triangle till you can develop it from memory.

2. Find the base of a triangle whose area is 5700 sq. ft. and an altitude of 76 ft.

3. Find the altitude of a triangle whose base is 40 rd. and whose area is 5 A.

4. Find the perpendicular of a right triangle when the base and hypotenuse are respectively 30 ft. and 50 ft.

5. Find the base when the perpendicular and hypotenuse are respectively 45 ft. and 90 ft.

6. Find the hypotenuse when the base and perpendicular are respectively 25 in. and 35 in.

7. Find the diagonal of a rectangle whose base and altitude are respectively 10 ft. and 36 ft.

8. Find the altitude of a rectangle whose base and diagonal are respectively 37 ft. and 90 ft.

9. A house measures 20 ft. from north to south; 15 ft. from east to west, and 12 ft. high. Find the distance on the floor from the northwest to the southeast corner.

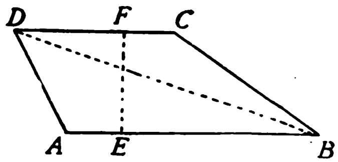
10. In the house described in No. 9, find the distance from the northwest lower corner to the southeast upper corner.

11. Commit to memory Relations IV and V.

125. The Other Quadrilaterals.—A **Trapezoid** is a quadrilateral, having only two parallel sides. The parallel sides are called *bases*.

A **Trapezium** is a quadrilateral, having no two of its sides parallel.

PROBLEM: *Develop the formula for the trapezoid.*



Development: Let $ABCD$ be a trapezoid; AB, *lower base* (B); DC, upper base (b); and FE, altitude. The diagonal BD divides the trapezoid into two triangles. In the triangle DAB, having for its base AB (B) and its altitude FE (a), we have—

$$\text{Area} = \frac{aB}{2}.$$

In the triangle BCD, having for its base DC (b) and for its altitude FE (a), we have—

$$\text{Area} = \frac{ab}{2}.$$

But the sum of the areas of the two triangles is the area of the trapezoid. Then,

$$A=\frac{aB}{2}+\frac{ab}{2}=\frac{aB+ab}{2}.$$

Factoring a out of both terms of the numerator of the fraction, we have the formula:

$$A=\frac{a(B+b)}{2}.$$

RELATION VI. *Abstractly, the area of a trapezoid is equal to the product of the altitude and the sum of the bases, divided by 2.*

There is no formula for a trapezium. If one of its diagonals and the altitudes from the other vertices upon that diagonal are known, its area may be easily found by the formula for triangles.

EXAMPLES.

1. Find the area of a trapezoid, if its altitude is 26 ft. and its bases are 50 ft. and 75 ft.

Solution: $A=\frac{26\ (50+75)}{2}=1625.$

$\therefore$ the reqd. area is 1625 sq. ft.

2. Find the upper base of a trapezoid whose lower base is 35 in., altitude 15 in., and area 360 sq. in.

Solution: (1) $\frac{15\ (35+b)}{2}=360.$

$2\times(1)=(2)$ $15\ (35+b)=720.$

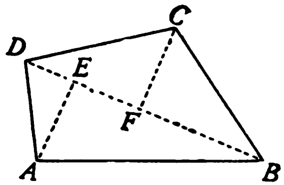
$\frac{1}{15}$ of $(2)=(3)$ $35+b=48.$

(4) $b=48-35=13.$

$\therefore$ the reqd. base is 13 in.

3. Find the area of a trapezium whose diagonal is 48 ft. and the altitudes upon that diagonal are respectively 10 ft. and 17 ft.

Explanation: Let $ABCD$ be the trapezium; DB, the diagonal; and EA and CF, altitudes. In the triangle DBC, we have given the base DB, and altitude CF; and in the triangle DAB, we have the base DB, and altitude EA.



SOLUTION.

1st triangle: $A = \frac{48 \times 17}{2} = 408.$

2d triangle: $A = \frac{48 \times 10}{2} = 240.$

$\therefore$ area of trapezium $= 408$ sq. ft. $+ 240$ sq. ft $=$ 648 sq. ft.

EXERCISE CI.

1. Define and draw a trapezoid; draw one diagonal and the altitude.

2. Define and draw a trapezium.

3. Find the area of a trapezoid whose bases are 46 yd. and 80 yd., and whose altitude is 36 yd.

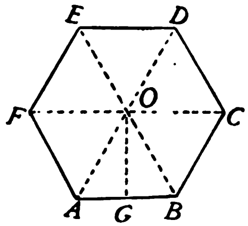
4. Find the altitude of a trapezoid whose bases are 30 ft. and 50 ft., and whose area is 480 sq. ft.

5. Find unknown base of a trapezoid whose altitude is 24 in., area 912 sq. in., and given base 45 in.

6. Find the area of a trapezium whose diagonal is 78 ft., and altitudes upon that diagonal are 43 ft. and 21 ft.

7. Study until you can solve from memory the problem of this Article.

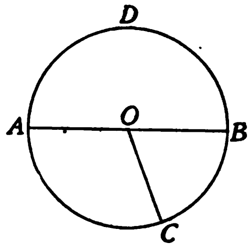
126. Regular Polygons and Circles.—A **Regular Polygon** has its sides equal and its angles equal. A regular polygon may always be separated into equal isosceles triangles by drawing lines from its center to each vertex. A line from the center, perpendicular to a side, is called the *apothem*. The lines bounding a polygon are called its *perimeter*.



OG, apothem.

NOTE.—A pentagon has 5 sides; a hexagon, 6 sides; a heptagon, 7 sides; an octagon, 8 sides; a decagon, 10 sides.

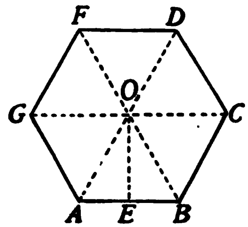
A **Circle** is a plane bounded by a curved line, all parts of which are equally distant from a point in the plane. The point is called the *center* and the curved line bounding the circle is called the *circumference.* A straight line, passing through the center of a circle and terminating at each end in the circumference, is called a *diameter.* A straight line extending from the center to the circumference is called a *radius.*



ADBC, a circumference; *AB*, a diameter; *OC*, a radius.

PROBLEM 1: *Develop the formula for the regular polygon.*

Development: Let $ABCDFG$ be a regular polygon. Divide it into equal triangles by drawing lines from the center to each vertex. Draw OE, the apothem, which is the length of the altitude of each triangle. Represent the bases of the triangles by b, b', b'', etc. Then,



In 1st triangle: $A = \frac{ab}{2}$

In 2d triangle: $A = \frac{ab'}{2}$

In 3d triangle: $A = \frac{ab''}{2}$, and so on.

The whole area $= \frac{ab+ab'+ab''+etc.}{2}$; or,

$$= \frac{a(b+b'+b''+etc.)}{2}.$$

But $b+b'+b''+etc.$ = the perimeter (P) of the polygon. Then, the formula for the polygon is—

$$A = \frac{aP}{2}.$$

RELATION VII. *Abstractly, the area of a regular polygon is equal to the product of its apothem and perimeter,* ***divided by 2.***

Relation VIII. *The ratio of the circumference (C) of a circle to its diameter (D) is 3.1416 (π).*

Note 1.—The character π is the Greek letter *pi*, and is used for 3.1416, in order to shorten the work in writing formulas.

Note 2.—The development of this relation is too difficult to be given here. It is found in geometry.

Formula:

$$\pi = \frac{C}{D} = 3.1416.$$

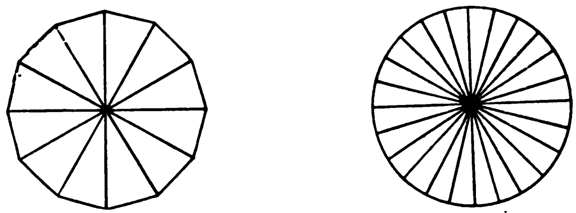
Since the diameter equals two times the radius (R), we may substitute $2R$ for D.

$$\frac{C}{2R} = \pi;\ \text{or,}$$

$$C = 2\pi R.$$

Relation IX. *The circumference of a circle is equal to 2 times 3.1416 times the radius.*

Problem 2: *Develop the formula for the area of the circle.*



Development: The figures above suggest the idea, that as the number of sides of a regular polygon increase, the polygon approaches the form of a circle. Proceeding upon the suggestion, and considering a circle a regular polygon of an infinite number of sides, we observe that the *apothem* becomes the *radius* of the circle and the *perimeter* becomes the *circumference* and the formula for the polygon

$$A = \frac{aP}{2},$$

becomes $A = \frac{RC}{2}$.

But from the last formula we know that the circumference equals $2\pi R$. Putting this in the place of C, we have—

$$A = \frac{2\pi RR}{2}\text{; or,}$$

$$A = \pi R^2.$$

RELATION X. *Abstractly, the area of a circle is equal to 3.1416 times the square of the radius.*

EXAMPLES.

1. Find the area of a regular pentagon whose apothem is 10 ft. and one side 14.53 ft.

Solution: (1) $P = 5 \times 14.53$ ft. $= 72.65$ ft.

(2) $A = \dfrac{10 \times 72.65}{2} = 363.25.$

$\therefore$ the area is 363.25 sq. ft.

2. Find the diameter of a circle whose circumference is 219.912 rods.

Solution: (1) $\dfrac{C}{D} = 3.1416.$

(2) $\dfrac{219.912}{D} = 3.1416.$

(3) $219.912 = 3.1416\,D.$

(4) $D = \dfrac{219.912}{3.1416} = 70.$

$\therefore$ the diameter is 70 rd.

3. Find the area of a circle whose radius is 200 ft.

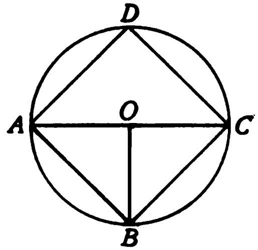
Solution: (1) $A = \pi R^2.$

(2) $A = 3.1416 \times 40000 = 125664.$

$\therefore$ the area is 125664 sq. ft.

4. Find the area of the square that can be inscribed in a circle whose diameter is 20 ft.

NOTE.—A square is inscribed in a circle, if all of its angles lie upon the circumference. A square is circumscribed about a circle, if it incloses the circle with its sides touching the circumference.



Explanation: In the figure, OB and OC are radii of the circle, and are therefore each 10 feet long. But these radii are sides of the right triangle COB, which is one-fourth of the required square $ABCD$. Find the area of the triangle and multiply by 4.

Solution: In the triangle: $A = \frac{10 \times 10}{2} = 50.$

$\therefore$ the area of square $= 4 \times 50$ sq. ft. $= 200$ sq. ft.

EXERCISE CII.

1. Study and give from memory the solutions of the two problems of this Article.

2. Commit to memory Relations VII to X.

3. Define and draw a regular polygon.

4. What is a pentagon? Octagon? Decagon?

5. Define a circle; circumference; diameter; radius; perimeter; apothem.

6. What is the formula for the area of a circle? A regular polygon?

7. Find diameter of a circle whose area is 2827.44 sq. in.

8. Find the radius of a circle whose circumference is 376.992 feet.

9. Find the apothem of a regular polygon whose area is 20800 sq. yd. and whose perimeter is 520 yd.

10. Find the area of a circle whose circumference is 37.6992 ft.

11. Upon how much pasture can a horse graze, if he is tied by a rope 60 ft. long?

12. A horse is tied to the corner of a house 40 ft. wide and 60 ft. long, by a rope 90 feet long. Over how much ground can the horse go?

13. Find one side of the square that can be inscribed in the circle whose diameter is 50 ft.

14. Find the area of the circle if one side of its inscribed square is 25 ft.

15. Find the circumference of a circle whose area is 254.4696 sq. ft.

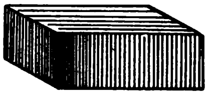
16. The radius of a circle is 34 inches. Find the area of the inscribed square.

17. A circle is 62.832 ft. in circumference. Find the area of the circumscribed square.

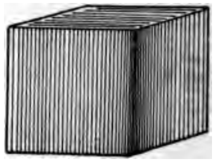
3. SOLIDS.

127. Polyhedrons.—Define solid. (See p. 156.)

A **Polyhedron** is a solid bounded by planes. The bounding planes are called *faces*. The lines of intersection of the planes are called *edges*. The points where the lines of intersection meet are called *vertices*.



A rectangular solid.

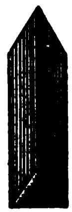


A cube.

A **Rectangular Solid** is a polyhedron bounded by rectangular planes.

A **Cube** is a rectangular solid whose dimensions are all equal.

A **Prism** is a polyhedron two of whose faces called *bases* are parallel polygons, and the other faces, called *lateral faces*, intersect in parallel lines. The altitude of a prism is the perpendicular distance between the planes of its bases. If all the lateral faces of a prism are rectangles, the prism is a *right prism*; if not, it is an *oblique prism*.



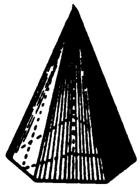
A right triangular prism.

NOTE 1.—We will study only right prisms in this book. The *lateral* (or *convex*) *surface* of a prism is the sum of the areas of the lateral faces;

the *total surface* is the sum of the lateral area plus the areas of the two bases.

NOTE 2.—A prism with a triangle for a base is a *triangular* prism; with a square base, a *square* prism; with a hexagon for a base, a *hexagonal* prism, etc.

A **Pyramid** is a polygon one of whose faces, the *base*, is a polygon, and the lateral faces triangles meeting in a point called the *vertex* of the pyramid. If the lateral faces are all equal *isosceles* triangles, the pyramid is called a *right* or *regular pyramid;* if the lateral faces are not *isosceles* triangles, the pyramid is an *oblique pyramid.* The perpendicular distance from the vertex to the base is called the *altitude.*



A right pentagonal pyramid.

The *slant height* of a right pyramid is the distance from the vertex to the middle point of one side of the base.

NOTE.—A pyramid with a triangle for a base is a *triangular* pyramid; with a square base, a *square* pyramid; with a pentagon for a base, a *pentagonal* pyramid, etc.

EXERCISE CIII.

Define:

1. A solid.
2. A polyhedron.
3. A face of a polyhedron.
4. An edge of a polyhedron.
5. A vertex of a polyhedron.
6. A rectangular solid.
7. A cube.
8. A prism.
9. Bases of a prism.
10. The altitude of a prism.
11. A right prism.
12. An oblique prism.
13. A triangular prism.
14. A square prism.
15. An octagonal prism.
16. A pyramid.
17. The altitude of a pyramid.
18. A triangular pyramid.
19. A pentagonal pyramid.
20. A right pyramid.
21. Lateral area.
22. Total area.

PROBLEM 1: *Letting* l, w *and* t *represent length, width and*

thickness respectively, develop the formula for the volume of a rectangular solid.

BY EQUATION.

Solution :

(1) Vol. of a solid l ft. l., w ft. w., t ft. th. = ( ) cu. ft.? (Question.)
(2) Vol. of a solid 1 ft. l., 1 ft. w., 1 ft. th. = 1 cu. ft. (Basis.)
$l \times (2) = (3)$ Vol. of a solid l ft. l., 1 ft. w., 1 ft. th. = l cu. ft.
$w \times (3) = (4)$ Vol. of a solid l ft. l., w ft. w., 1 ft. th. = lw cu. ft.
$t \times (4) = (5)$ Vol. of a solid l ft. l., w ft. w., t ft. th. = lwt cu. ft., answer.

BY PROPORTION.

Solution :

(1) Vol. of a solid l ft. l., w ft. w., t ft. th. = ( ) cu. ft.? (Question.)
(2) Vol. of a solid 1 ft. l., 1 ft. w., 1 ft. th. = 1 cu. ft. (Basis.)
(3) $l \times w \times t : 1 \times 1 \times 1 :: ( ) : 1$?
(4) $\dfrac{l \times w \times t \times 1}{1 \times 1 \times 1} = lwt$, answer.

Using V for *volume*, the result may be written as follows:

$$V = lwt.$$

RELATION XI. *Abstractly, the volume of a rectangular solid is equal to the continued product of its length, width and thickness.*

PROBLEM 2: *Develop the formula for the volume of a cube.*

Development : Since a cube is a rectangular solid whose length, width, and thickness are all equal, each dimension in the last formula may be represented by E (edge). Then, the formula may be written —

$$V = E \times E \times E; \text{ or,}$$

$$V = E^3.$$

RELATION XII. *Abstractly, the volume of a cube is equal to the cube of its edge.*

PROBLEM 3: *Develop the formula for the lateral or convex area of a right prism.*

Development: Since the lateral faces of a right prism are rectangles all having one common altitude, the altitude of the prism, and the sum of their bases the perimeter of the base of the prism, the sum of their areas will be the product of these; or,

$$\text{L. A.} = Pa.$$

RELATION XIII. *Abstractly, the lateral area of a right prism is equal to the product of the altitude and the perimeter of the base.*

PROBLEM 4: *Develop the formula for the total area of a right prism.*

Development: If the area of each base be represented by B, the sum of the areas of the two bases added to the lateral area will give the total area; or,

$$\text{T. A.} = aP + 2B.$$

RELATION XIV. *Abstractly, the total area of a right prism is equal to the product of the altitude and perimeter of the base, plus two times the area of the base.*

NOTE.—Remember that the area of the bases of a prism will have to be found by the formulas for finding the areas of polygons.

PROBLEM 5: *Develop the formula for the volume of a right prism, letting* B *represent the area of the base and* **a** *the altitude.*

Solution: (1) Vol. of a prism with B sq. ft. base, a ft. high = ( ) cu. ft.? (Question.)

(2) Vol. of a prism with 1 sq. ft. base, 1 ft. high = 1 cu. ft. (Basis.)

$B \times (2) = (3)$ Vol. of a prism with B sq. ft. base, 1 ft. high = B cu. ft.

$a \times (3) = (4)$ Vol. of a prism with B sq. ft. base, a ft. high = Ba cu. ft., answer.

Using V for volume, the formula may be written—

$$V = Ba.$$

RELATION XV. *Abstractly, the volume of a right prism is equal to the product of the area of the base and the altitude.*

NOTE.—Have the pupil solve the above by proportion.

PROBLEM 6: *Develop the formula for the lateral area of a right pyramid.*

Development: In a right pyramid, the lateral area is composed of equal isosceles triangles whose bases together form the *perimeter* (P) of the base of the pyramid, and whose common altitude is the *slant height* (s) of the pyramid. Therefore,

$$\text{L. A.} = \frac{Ps}{2}.$$

RELATION XVI. *Abstractly, the lateral area of a right pyramid is equal to the product of the perimeter of the base and the slant height, divided by 2.*

PROBLEM 7: *Develop the formula for the total area of a right pyramid.*

Development: If the area of the base (B) be added to the lateral area, the sum will be the total area, or,

$$\text{T. A.}=\frac{Ps}{2}+B.$$

RELATION XVII. *Abstractly, the total area of a right pyramid is equal to the product of the perimeter of the base and the slant height, divided by 2, plus the area of the base.*

RELATION XVIII. *Abstractly, the volume of a right pyramid is equal to the product of the area of the base and the altitude, divided by 3.*

NOTE.—The development of this relation is too difficult to be given before the pupil has studied geometry. It may be expressed in the following formula:

$$V=\frac{Ba}{3}.$$

EXERCISE CIV.

1. Study until you can solve from memory each problem of this Article.

2. Commit to memory Relations XI to XVIII.

3. Find the area of a rectangular solid 15 ft. long, 12 ft. wide, and $1\frac{1}{3}$ ft. thick.

4. Find the length of a solid whose volume is 1800 cu. in., width 30 in., thickness 12 in.

5. Find the volume of a cube, one of whose edges is 46 in.

6. Find one edge of a cube whose volume is 39304 cu. in.

7. Find the volume of a cube whose total area is 486 sq. in.

8. Find the lateral area and total area of a square prism, each side of the base being 7 in. and the altitude 48 in.

9. Find the volume of the prism in No. 8.

10. One side of the base of a square prism is 9 in., its volume is 1620 cu. in. Find its total area.

11. A right prism has for its base a right triangle whose perpendicular and hypotenuse are respectively 12 in. and 15 in., and the altitude of the prism is 30 in. Find the lateral area, total area, and the volume.

12. A square pyramid has for one side of its base 48 in. and its altitude 128 in. Find the slant height, the lateral area, total area, and volume.

13. A hexagonal pyramid has for one side of its base 12 in., the apothem of the base 10.4 in. If the volume is 7488 cu. in., find the altitude, slant height, total area.

14. Find the distance from one lower corner to the diagonal upper corner of a cube whose edge is 20 ft. (See Exercise C, No. 10.)

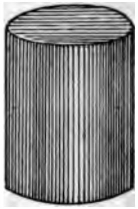
128. Solids Having Curved Surfaces.—A Curved Surface is a surface one or both of whose dimensions change direction at every point.

A **Globe** or **Sphere** is a solid bounded by a curved surface all parts of which are equally distant from a point within called the *center*. A straight line passing through the center of a sphere and terminating at each end in the surface is a *diameter* of the sphere. A straight line extending from the center to the surface is a *radius* of the sphere.

A sphere.

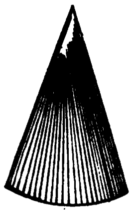
A *cylindrical surface* is a surface which uniformly and continuously changes the direction of one and only one of its dimensions.

A **Right Cylinder** is a solid bounded by two equal parallel circles and a cylindrical surface joining the circumferences of these circles at right angles to the planes of the circles. The circles are called *bases*, and the perpendicular distance between the bases, the *altitude*.



A right cylinder.

If, at the center of a circle, a perpendicular to the circle be erected, and lines could be drawn from some point in the perpendicular to every point in the circumference of the circle, the lines thus drawn would form a continuous curved surface. Such a surface is called a *conical surface*. A solid thus bounded by a circle and a conical surface is called a **Right Cone.** The circle is called the *base;* the point at the top, the *apex;* the perpendicular from the apex to the base, the *altitude;* and a straight line from the apex to the circumference of the base, the *slant height*.



A right cone.

NOTE.—The author has found it difficult to frame definitions for *cylinder* and *cone* sufficiently accurate to satisfy him and at the same time sufficiently simple in its wording to be easily comprehended by the average pupil. The teacher should be especially careful in explaining these definitions.

RELATION XIX. *Abstractly, the area of the surface of a sphere is equal to 4 times 3.1416 times the square of the radius of the sphere.*

NOTE.—The development of this relation is too difficult to be given before the pupil has studied geometry. It may be expressed in the following formula:

$$A = 4\pi R^2.$$

Notice that this area is just *4 times* the area of a circle having the same radius.

PROBLEM 1: *Develop the formula for the volume of a sphere.*

Development: A sphere is sometimes thought of as made up of an infinite number of pyramids, all having their vertices at the center of the sphere; the radius of the sphere for their *altitudes;* and their bases so infinitely small as to form one continuous surface—the surface of the sphere. Thus considered, the volume of the sphere is the sum of the volumes of all these pyramids, the sum of whose bases is the area (A) of the surface of the sphere, and whose common altitude is the radius (R) of the sphere. Then,

$$V=\frac{AR}{3}.$$

But from relation XIX,

$$A=4\pi R^2.$$

Then,

$$V=\frac{AR}{3}=\frac{4\pi R^2R}{3}; \text{ or,}$$

$$V=\frac{4\pi R^3}{3}.$$

RELATION XX. *Abstractly, the volume of a sphere is equal to 4 times 3.1416 times the cube of the radius, divided by 3.*

PROBLEM 2: *Develop the formulas for the right cylinder.*

Development: If we may regard the circle as a regular polygon of an infinite number of sides (see problem 2, p. 229), then, the cylinder may be considered as a regular prism with an infinite number of lateral faces so small as to form a curved surface. Then, all the formulas for the right prism are true for the right cylinder.

$$\text{L. A.}=Pa.$$
$$\text{T. A.}=Pa+2B.$$
$$V=Ba.$$

NOTE.—In the cylinder, the base (B) is the area of the circle and the perimeter (P) is the circumference of the circle.

RELATIONS: *See those for the prism* (Relations XIII–XV).

PROBLEM 3. *Develop the formulas for the right cone.*

Development: The same line of reasoning that makes the cylinder a prism with an infinite number of faces, makes the cone a regular pyra-

3. Two men buy a grindstone 2 ft. in diameter; the first grinds till the stone is 18 in. in diameter. How much should each pay, allowing a circular hole in the center 3 in. in diameter, which, if it were counted as stone, would be worth 2¢?

4. A and B buy a ball of twine 6 in. in diameter for 16¢. A winds off the twine till the ball is but 3 in. in diameter, and then gives the ball to B. How much should each pay?

5. How many leaden balls $\frac{1}{2}$ in. in diameter can be made from a leaden ball 2 in. in diameter?

6. The diameter of Jupiter is 11 times that of the earth: (1) Its surface is how many times that of the earth? (2) Its volume is how many times that of the earth?

130. Review.—

EXERCISE CVII.

1. Find the hypotenuse of a right triangle, if its base and perpendicular are 36 ft. and 48 ft., respectively.

2. Find the base of a right triangle, if the hypotenuse and perpendicular are 100 ft. and 80 ft., respectively.

3. The base and perpendicular of a right triangle are equal, and the hypotenuse is 125 ft.: find the perpendicular.

4. Find the length of one side of the square that can be cut from a circle whose diameter is 250 inches.

5. How long is the ladder that can stand 27 ft. from a wall 36 ft. high, and reach the top of it?

6. Find the area of a triangle whose base and altitude are 180 ft. and 75 ft., respectively.

7. Find the area of a right triangle whose base and perpendicular are 30 ft. and 40 ft., respectively.

NOTE.—In a right triangle the base and perpendicular may be considered the base and altitude.

8. The area of a right triangle, whose base and perpendicular are equal, is 1800 sq. ft.: find the length of the base.

9. The area of a triangle is 1200 sq. ft., its altitude is 30 ft.: find the length of its base.

10. Find the area of the inner surface of the walls of a house, if the dimensions are 30 ft. by 20 ft. by 12 ft. in the clear.

11. Subtract from the result found in No. 10 the area of 6 windows 6 ft. by 2 ft. 3 in., and 3 doors 6 ft. 8 in. by 3 ft. 4 in.

12. A field is in the shape of a parallelogram whose base is 50 rd. and whose altitude is 20 rd.: find how many acres it contains.

13. A rectangular field has one side twice as long as the other; its area is 20 acres: find the length of its sides.

NOTE.—Reduce the acres to square rods.

14. The length of a rectangular field is to its width as 8 to 5, and it contains 25 acres: find its sides.

15. A trapezoid has for its bases 49 ft. and 71 ft., and its altitude is 50 ft.: find its area.

16. A pasture, in the form of a trapezoid, contains 1200 acres, its parallel sides are 2 mi. and 3 mi. long: how far across it?

17. In a trapezoid the bases are to each other as 2 to 3, the altitude is 14 ft., and the area is 350 sq. ft.: find the length of the bases.

18. The circumference of a circle is 326.72536 ft.: find the radius, diameter, and area.

19. The area of a circle is 45238.896 sq. ft.: find the radius and circumference.

20. How long must a rope be to fasten a horse so that he may graze upon ½ acre and no more?

21. Two sides of a rectangle are 30 ft. and 40 ft.: find the area of the circumscribed circle.

NOTE.—The diameter of the circle is the diagonal of the rectangle.

22. A circle is circumscribed about a rectangle whose sides are to each other as 3 to 4: find the area of the rectangle if the diameter of the circle is 60 ft.

23. Find the number of cubic inches in a bushel measure, a right cylinder whose altitude is 8 inches and the radius of whose base is $9\frac{1}{4}$ inches.

24. What will it cost to paint the convex surface of a cylindrical tank 8 ft. high and 14 ft. across the bottom, if the painter charges $15\frac{1}{2}$¢ per sq. yd.?

25. How many gallons of water will the tank described in No. 24 hold? (Consider the dimensions inside measure.)

26. Find the volume of a rectangular solid, if its base is a rectangle 10 in. by 25 in. and its altitude 20 in.

27. Find the surface of the solid in No. 26.

28. Find the surface and the volume of a cube, if its altitude is 17 ft.

29. The apothem of the base of a right hexagonal pyramid is 8 in., the length of one side of the base is 9.2376 in., and the altitude of the pyramid is 3 ft. Find the volume.

30. Find the whole surface of the pyramid in No. 29.

31. Find the volume of a right cone the diameter of whose base is 18 ft. and whose altitude is 30 ft.

32. Find the surface of the cone in No. 31.

33. The circumference of a sphere is 188.496 in.: find its volume and surface.

34. The diameter of a sphere is 48 in.: find its surface and volume.

35. The volume of a sphere is 1767.15 cu. in.: find its radius, diameter, circumference, and surface.

36. The surface of the earth is 196663355.7504 sq. mi.: find its volume.

37. The bases of two similar triangles are respectively 10 in. and 16 in. If the altitude of the smaller is 9 ft. 2 in., find the altitude of the larger.

B. PERCENTAGE.

1. THE PERCENTAGE FORMULA.

131. Elements.—There are three elements in percentage problems:

(1) *A number some per cent of which is to be considered.* This number is called the *Base;* and, in the percentage formula to follow, it is represented by B.

(2) *A number of per cent of the base.* This number* is represented in the formula by p.

(3) *The number which equals the per cent of the base considered.* This number is called the *percentage;* and, in the formula, is represented by P.

To illustrate:

5% of 240 is 12.

5 is the number of per cent, p.
240, is the base, B.
12 is the percentage, P.

Many authors speak of two additional elements, *Amount* and *Difference.* For example:

25% more than cost = 100% of cost + 25% of cost. (Amount.)
25% less than cost = 100% of cost − 25% of cost. (Difference.)

It is not necessary to put any special study or emphasis upon these as elements.

25% more than a number = 125% of that number.
25% less than a number = 75% of that number.

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*This number has received the name *Rate*, and in the formulas of most arithmetics is represented by R. It is probable that it has been so called from the fact that this number is often a rate. For example: *Rates* of interest, *rates* of commission, *rates* of gain or loss, *rates* of insurance, are usually expressed in *per cent.* But when applied to the simple percentage problem, the word *rate* is a *misnomer.* One number may be some *per cent* of another, but not some *rate* of another.

EXERCISE CVIII.

Recite this exercise orally:

1. 50% more than cost=( )% of cost?
2. 94% more than my age=( )% of my age?
3. 47% less than $500=( )% of $500?
4. 35% less than the price=( )% of the price?
5. 143% of $900=( )% more than $900?
6. $66\frac{2}{3}$% of $600=( )% less than $600?
7. 100% more than a number=( )% of that number?
8. 100% less than a number=( )% of that number?
9. 500% of a number=( )% more than that number?

132. Developing the Percentage Formula.—The *percentage formula* is the equation which relates the three elements, *base*, *per cent*, and *percentage*. It may be developed either by the equation method or the proportion method of solution.

PROBLEM: *Find p% of B.*

NOTE.—We know that 100% of anything is all of it; then, we know that 100% of $B = B$.

BY EQUATIONS.

Solution: (1) $p\%$ of $B = (\ )$? (Question.)
(2) 100% of $B = B$. (Basis.)

$\frac{1}{100}$ of (2) = (3) 1% of $B = \frac{B}{100}$.

$p \times (3) = $ (4) $p\%$ of $B = \frac{pB}{100}$, answer.

BY PROPORTION.

Solution: (1) $p\%$ of $B = (\ )$? (Question.)
(2) 100% of $B = B$. (Basis.)
(3) $p : 100 :: (\ ) : B$?
(4) $\frac{p \times B}{100} = \frac{pB}{100}$, answer.

But, since this answer represents the percentage, P, we have—

$$P = \frac{pB}{100}.$$

RELATION: *Abstractly, the percentage is equal to the product of the base and the number of per cent, divided by 100.*

EXAMPLES.

1. Find 8% of \$50.

Solution: (1) $P=\frac{pB}{100}$.

(2) $P=\frac{8\times50}{100}=4$.

Answer, \$4.

2. \$20.80 is 4% of what number?

Solution: (1) $\frac{4B}{100}=20.8$. (Why?)

(2) $B=\frac{20.8\times100}{4}=520$.

Answer, \$520.

3. What % of \$600 is \$540?

Solution: (1) $\frac{600p}{100}=540$. (Why?)

(2) $p=\frac{540\times100}{600}=90$.

Answer, 90%.

4. 600 is 50% more than what number?

NOTE.—50% more than a number is 150% of that number.

Solution: (1) $\frac{150B}{100}=600$.

(2) $B=\frac{600\times100}{150}=400$, answer.

5. 520 is how many % less than 650?

Solution: (1) $\frac{650p}{100}=520$.

(2) $p=\frac{520\times100}{650}=80$.

Answer, 20% less. (Why?)

6. \$600 is 20% less than what number?

NOTE.—20% less than a number is 80% of it.

Solution: (1) $\frac{80\,B}{100}=600.$

(2) $B=\frac{600\times 100}{80}=750.$

Answer, \$750.

7. Find a number B, if $p\%$ of it is P. (By the equation method.)

Solution: (1) 100% of $B=(\ )$? (Question.)

(2) $p\%$ of $B=P$. (Basis.)

$(2)\div p=(3)$ 1% of $B=\frac{P}{p}.$

$100\times(3)=(4)$ 100% of $B=\frac{100\,P}{p}$; or,

$B=\frac{100\,P}{p}$, result.

NOTE.—Let the pupil give the relation.

EXERCISE CIX.

1. Commit to memory the relation given on p. 249.

2. Develop from memory the percentage formula.

3. Show how formula in example 7 can be obtained directly from the percentage formula.

4. P is how many % of B?

NOTE.—The solution of number 4 gives the formula $p=\frac{100\,P}{B}$

5. Show that $p=\frac{100\,P}{B}$ can be obtained directly from the percentage formula.

6. \$54 is 9% of what?

7. Find 47% of \$460.

8. 56 is what % of 64?

9. 564 is 20% more than what?
10. 376 is 20% less than what?
11. What number added to 20% of itself gives 564?

NOTE.—Show that Nos. 9 and 11 are the same.

12. What number less 20% of itself gives 376?

NOTE.— Show that Nos. 10 and 12 are the same.

13. What number added to $16\frac{2}{3}$% of itself gives 770?
14. What number less 58% of itself gives 105?
15. \$360 is how many % of \$600?
16. $\frac{3}{4}$ is how many % more than $\frac{1}{2}$?
17. $\frac{1}{2}$ is how many % less than $\frac{3}{4}$?
18. 5% of 420 is how many % less than 35?
19. \$35 is how many % more than 7% of \$400?
20. 40% of \$500 is how many % of $\frac{3}{4}$ of \$800?

2. PERCENTAGE WITHOUT TIME.

133. Profit and Loss.

PRINCIPLES: *1. The gain or loss is some number of per cent of the cost price.*

2. The selling price is equal to the cost price plus the gain or minus the loss.

TERMS.

C, cost price.	L, Loss.
S, selling price.	p, number of % of gain or loss.
G, gain or profit.	

Development of formulas for Profit and Loss:

1. $G = \frac{pC}{100}$. (Prin. 1.)

2. $L = \frac{pC}{100}$. (Prin. 1.)

RELATION I. *Abstractly, the gain or loss is equal to the product of cost price and the number of per cent of gain or loss, divided by 100.*

When there is a gain,

(1) $S=C+G$. (Prin. 2.)

But, (2) $G=\frac{pC}{100}$

Then, (3) $S=C+\frac{pC}{100}=\frac{100C+pC}{100}$; or,

$$3.\quad S=\frac{C\,(100+p)}{100}$$

When there is a loss,

(1) $S=C-L$. (Prin. 2.)

But, (2) $L=\frac{pC}{100}$.

Then, (3) $S=C-\frac{pC}{100}=\frac{100\,C-pC}{100}$; or,

$$4.\quad S=\frac{C\,(100-p)}{100}.$$

RELATION II. *Abstractly, the selling price is equal to the product of the cost price and 100 plus the per cent of gain, divided by 100; or the product of the cost price and 100 minus the per cent of loss, divided by 100.*

EXAMPLES.

1. How much does a man gain by selling an article that cost $12.50 at 40% profit?

Solution: $G=\frac{40\times 12.5}{100}=5$. (Why?)

Answer, $5.

2. An article cost $4.20: for how much must it be sold to gain 20%?

Solution: $S=\frac{4.2\times 120}{100}=5.04$. (Why?)

Answer, $5.04.

3. A drover paid \$1320 for cattle that he afterwards sold at a loss of $16\frac{2}{3}\%$. What did he lose?

Solution: $L = \frac{16\frac{2}{3} \times 1320}{100} = 220.$ (Why?)

Answer, \$220.

NOTE.—Use cancellation when practicable. $16\frac{2}{3}$ is contained in 100 6 times.

4. Flour that cost \$4.50 per barrel sold for \$4.95 per barrel. What was the per cent of gain?

Solution: (1) $S = \frac{C(100+p)}{100}.$

(2) $4.95 = \frac{4.5\,(100+p)}{100}.$

(2) = (3) $\frac{4.5\,(100+p)}{100} = 4.95.$

100 × (3) = (4) $450 + 4.5\,p = 495.$

Transpose, (5) $4.5\,p = 495 - 450 = 45.$

(5) ÷ 4.5 = (6) $p = 10.$

Answer, 10%.

Another Solution: (1) $G = S - C = \$4.95 - \$4.50 = \$.45.$

(2) $\frac{Cp}{100} = G.$

(3) $\frac{4.5\,p}{100} = .45.$

100 × (3) = (4) $4.5\,p = 45.$

(4) ÷ 4.5 = (5) $p = 10.$

Answer, 10%.

5. Find the per cent of loss, if goods that cost \$2720 sell for \$2380.

Solution: (1) $S = \frac{C(100-p)}{100}.$

(2) $2380 = \frac{2720\,(100-p)}{100}.$

100 × (2) = (3) $238000 = 272000 - 2720\,p.$

Transpose, (4) $2720\,p = 272000 - 238000 = 34000.$

(4) ÷ 2720 = (5) $p = \frac{34000}{2720} = 12\frac{1}{2}.$

Answer, $12\frac{1}{2}\%$.

Another Solution: (1) $L = C - S = \$2720 - \$2380 = \$340.$

(2) $\frac{Cp}{100} = L.$

(3) $\frac{2720\,p}{100} = 340.$

(4) $2720\,p = 34000.$

(5) $p = \frac{34000}{2720} = 12\frac{1}{2}.$

Answer, $12\frac{1}{2}\%$.

NOTE.—In No. 4, first solution, equation (2) turned around gives equation (3). If I had transposed, I would have a minus sign before the p. In No. 5, first solution, I transposed in equation (3) to get equation (4). If I had turned the equation around as in No. 4, I would have a minus sign before the p. *If the letter has a plus sign in the right member, turn the equation around, to* ***change to the left member; if the letter has a minus sign, transpose,*** to get rid of the minus sign.

6. A merchant bought goods at 20% less than market value and sold them at 20% more than market value. What % did he gain?

Solution: (1) $C = 80\%$ of market value. (Why?)
(2) $G = 40\%$ of market value. (Why?)

NOTE.—The problem then becomes, "40% is what % of 80%?"

(3) $\frac{Cp}{100} = G.$

(4) $\frac{80\,p}{100} = 40.$

(5) $p = \frac{40 \times 100}{80} = 50.$

Answer, 50%.

Oral Solution: Since the cost is 80% of market value, and the gain is 40% of the market value, the gain is one-half, or 50% of the cost.

7. A man lost $16\frac{2}{3}\%$ by selling a house for \$538 less than it cost. Find the cost.

Solution: (1) $L = \dfrac{pC}{100}$.

(2) $538 = \dfrac{16\frac{2}{3}C}{100}$.

(3) $C = \dfrac{538 \times 100}{16\frac{2}{3}} = 3228$.

Answer, $3228.

8. A merchant sold goods for $980.28, thereby losing 10%. Find the cost.

Solution: (1) $S = \dfrac{C\,(100-p)}{100}$.

(2) $980.28 = \dfrac{C\,(100-10)}{100} = \dfrac{90\,C}{100}$.

(3) $C = \dfrac{980.28 \times 100}{90} = 1089.2$.

Answer, $1089.20.

9. How must I mark a pair of shoes that cost $2.40, to gain 25%? (Key: *Come and buy.*)

Solution: $S = \dfrac{2.4\,(100+25)}{100} = 3$.

Answer, *m.yy*.

NOTE.—*Keys* are employed by many merchants to mark the selling price as well as the cost price of their goods, in a manner known to themselves only. The letters of the key, taken in order, represent the figures:

c	o	m	e	a	n	d	b	u	y
1	2	3	4	5	6	7	8	9	0

Then, $3.00 = *m.yy*. A separate character, as *x*, is often used instead of repeating a letter. Thus *m.yy* may be written *m.yx*.

EXERCISE CX.

1. What is the base in profit-and-loss problems?

2. Commit to memory relations I and II.

3. Give from memory the formulas under relation I.

4. Develop all the formulas for profit and loss.

6. \$600 is 20% less than what number?

NOTE.—20% less than a number is 80% of it.

$$\text{Solution: (1) } \frac{80\,B}{100}=600.$$

$$\text{(2) } B=\frac{600\times 100}{80}=750.$$

Answer, \$750.

7. Find a number B, if $p\%$ of it is P. **(By the equation method.)**

Solution: (1) 100% of $B=(\ )$? (Question.)
(2) $p\%$ of $B=P$. (Basis.)

$$(2)\div p=(3)\ 1\%\text{ of }B=\frac{P}{p}.$$

$$100\times(3)=(4)\ 100\%\text{ of }B=\frac{100\,P}{p};\text{ or,}$$

$$B=\frac{100\,P}{p},\text{ result.}$$

NOTE.—Let the pupil give the relation.

EXERCISE CIX.

1. Commit to memory the relation given on p. 249.
2. Develop from memory the percentage formula.
3. Show how formula in example 7 can be obtained directly from the percentage formula.
4. P is how many % of B?

NOTE.—The solution of number 4 gives the formula $p=\frac{100\,P}{B}$

5. Show that $p=\frac{100\,P}{B}$ can be obtained directly from the percentage formula.

6. \$54 is 9% of what?
7. Find 47% of \$460.
8. 56 is what % of 64?

...h bought goods and sold them at a loss of $12\frac{1}{2}\%$, ...l the selling price.

...a horse, and afterwards sold him at a loss of ... 20% to what I then had, and with this amount ...her horse. I sold the second horse at a profit of ...44. What did the first horse cost?

... Trade Discount.—Wholesale merchants usually ... the selling price of their goods so that they can make a ...uction from this price in selling to their customers. The ...*arked price* is often spoken of as *list price*. The reduction ...rom the list price is called **Trade Discount.**

NOTE.—Discounts are made for one or more of the following reasons: (1) The goods may be marked in the first place with the expectation of making a uniform reduction to all retail customers, (2) for large purchasers, and (3) for cash.

"25% off," or "25 off," means 25% less than the list price. "25 and 10 off" means 25% less than list price, and then 10% less than the amount left after the first reduction. "$\frac{1}{3}$ and 5 off," means the same as "$33\frac{1}{3}$ and 5 off."

PRINCIPLES: *1. The first discount is some number of per cent of the list price.*

2. Each succeeding reduction, or discount, is some number of per cent of the amount left after making the last preceding reduction.

3. The selling price, after any discount, is equal to the price before the discount, minus the discount.

TERMS.

L, list price.
D, discount.
S, S', S'', etc., successive selling prices.
p, number of % off.

No.	*$ C*		*p%*	*$ G*		*$ L*		*$ S*	
5.	484		25	?				?	
6.	564		25			?		?	
7.	840		$33\frac{1}{3}$	?				?	
8.	420		$16\frac{2}{3}$			?		?	
9.	248	80	$12\frac{1}{2}$	?				?	
10.	590		?	59				?	
11.	730		?			219		?	
12.	48	50	?	?				58	20
13.	32	40	?			?		25	92
14.	?		?	8	58			65	78
15.	?		?			2	50	10	
16.	?		$18\frac{3}{4}$	6	30			?	
17.	?		$6\frac{1}{4}$			27	30	?	
18.	?		120	192				?	

19. Jones sells goods costing him $1500 at a profit of $16\frac{2}{3}\%$. Find the gain.

20. I sold 100 hogs, averaging 300 lb. each, at a profit of $33\frac{1}{3}\%$. If they cost me $1200, find the selling price per lb.

21. What was the loss, if a pair of shoes that cost $2.50 sold at 20% loss?

22. How would you mark an article that cost $3.20 to sell at a 20% profit? (Key: *Buy for cash.*)

23. If $187.50 is gained on goods that cost $1250, find the selling price, and per cent of gain.

24. If $260 is lost on goods that cost $1300, find the selling price and per cent of loss.

25. If goods sold at $16\frac{2}{3}\%$ profit for $1085, find the cost price.

26. If goods sold at 25% loss for $480, find the cost price.

27. Sold a horse for $80, losing 20%; with the $80 bought another, and sold him, gaining 20%. Find the loss on the two transactions.

28. A, failing in business, paid B $1512, which is only 60¢ on the $1 (60%). How much did A owe B?

29. Mr. Smith bought goods and sold them at a loss of $12\frac{1}{2}\%$, losing \$137. Find the selling price.

30. I bought a horse, and afterwards sold him at a loss of 20%; I added 20% to what I then had, and with this amount bought another horse. I sold the second horse at a profit of 20% for \$144. What did the first horse cost?

134. Trade Discount.—Wholesale merchants usually mark the selling price of their goods so that they can make a reduction from this price in selling to their customers. The *marked price* is often spoken of as *list price*. The reduction from the list price is called **Trade Discount.**

NOTE.—Discounts are made for one or more of the following reasons: (1) The goods may be marked in the first place with the expectation of making a uniform reduction to all retail customers, (2) for large purchasers, and (3) for cash.

"25% off," or "25 off," means 25% less than the list price. "25 and 10 off" means 25% less than list price, and then 10% less than the amount left after the first reduction. "$\frac{1}{3}$ and 5 off," means the same as "$33\frac{1}{3}$ and 5 off."

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3. The selling price, after any discount, is equal to the price before the discount, minus the discount.

TERMS.

L, list price.
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S, S', S'', etc., successive selling prices.
p, number of % off.

No.	$\$ C$		$p\%$	$\$ G$		$\$ L$		$\$ S$	
5.	484		25	?				?	
6.	564		25			?		?	
7.	840		$33\frac{1}{3}$	?				?	
8.	420		$16\frac{2}{3}$			?		?	
9.	248	80	$12\frac{1}{2}$	?				?	
10.	590		?	59				?	
11.	730		?			219		?	
12.	48	50	?	?				58	20
13.	32	40	?			?		25	92
14.	?		?	8	58			65	78
15.	?		?			2	50	10	
16.	?		$18\frac{3}{4}$	6	30			?	
17.	?		$6\frac{1}{4}$			27	30	?	
18.	?		120	192				?	

19. Jones sells goods costing him \$1500 at a profit of $16\frac{2}{3}\%$. Find the gain.

20. I sold 100 hogs, averaging 300 lb. each, at a profit of $33\frac{1}{3}\%$. If they cost me \$1200, find the selling price per lb.

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22. How would you mark an article that cost \$3.20 to sell at a 20% profit? (Key: *Buy for cash.*)

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PRINCIPLES: *1. The first discount is some number of per cent of the list price.*

2. Each succeeding reduction, or discount, is some number of per cent of the amount left after making the last preceding reduction.

3. The selling price, after any discount, is equal to the price before the discount, minus the discount.

TERMS.

L, list price.
D, discount.
S, S', S'', etc., successive selling prices.
p, number of % off.

Development of the formulas for Trade Discount.

$$1.\quad D=\frac{pL}{100}.\quad (\text{Prin. 1.})$$

RELATION I. *Abstractly, the discount is equal to the product of the list price and the number of per cent of discount, divided by 100.*

$$(1)\quad S=L-D.\quad (\text{Prin. 3.})$$

$$(2)\quad S=L-\frac{pL}{100}=\frac{100L-pL}{100};\ \text{or,}$$

$$2.\quad S=\frac{L(100-p)}{100}.$$

RELATION II. *When there is one reduction, abstractly, the selling price is equal to the product of the list price and 100 minus the number of per cent of reduction, divided by 100.*

When there are several reductions, the amount left, or S, of the first reduction becomes the L for the next reduction, and so on.

Then, if another per cent (p') be taken off the new selling price,

$$S'=\frac{S(100-p')}{100}.\quad (\text{Prin. 2.})$$

Putting in place of S its value from the formula above, we have:

$$3.\quad S'=\frac{L(100-p)(100-p')}{100\times 100}.$$

In the same way, a formula for any number of reductions may be obtained.

RELATION III. *After any number of reductions, abstractly, the selling price is equal to the fraction whose numerator is the product of the list price and the several remainders found by subtracting each number of per cent from 100, and whose denominator is the product of as many 100's as there are reductions.*

EXAMPLES.

1. Find a discount of 20% off of a list price of $540.

Solution: $D = \dfrac{20 \times 540}{100} = 108.$

Answer, $108.

2. If the list price is $460, find the selling price at 30 off.

Solution: $S = \dfrac{460 \times 70}{100} = 322.$ (Why ?)

Answer, $322.

3. If the list price is $500, find the selling price at 20 and 10 off.

Solution: $S' = \dfrac{500 \times 80 \times 90}{100 \times 100} = 360.$

Answer, $360.

4. I sold an article that cost me $12, at $\frac{1}{3}$, 15, and 5 off. Find the selling price.

Solution: $S' = \dfrac{12 \times 66\frac{2}{3} \times 85 \times 95}{100 \times 100 \times 100} = 6.46.$

Answer, $6.46.

NOTE.—$33\frac{1}{3}$ will cancel $66\frac{2}{3}$ and 100.

5. If an article sells at 30 off for $14, find the list price.

Solution: (1) $\dfrac{L(100 - p)}{100} = S.$

(2) $\dfrac{70\,L}{100} = 14.$

(3) $L = \dfrac{14 \times 100}{70} = 20.$

Answer, $20.

6. If an article sells at 30 and 10 off for $6.30, find the list price.

Solution: (1) $\dfrac{L(100 - p)(100 - p')}{100 \times 100} = S.$

(2) $\dfrac{70 \times 90 \times L}{100 \times 100} = 6.30.$

(3) $L = \dfrac{6.30 \times 100 \times 100}{70 \times 90} = 10.$

Answer, $10.

7. An article, listed at \$20, sold at ( ) and 10 off for \$13.50. Fill the blank.

$$\begin{aligned}
\textit{Solution:}\ (1)\ & 13.50=\frac{20\,(100-p)\times 90}{100\times 100}.\\
10000\times(1)=(2)\ & 135000=1800(100-p).\\
(3)\ & 135000=180000-1800\,p.\\
\text{Transpose, }(4)\ & 1800\,p=180000-135000.\\
(5)\ & 1800\,p=45000.\\
(6)\ & p=\frac{45000}{1800}=25.
\end{aligned}$$

Answer, 25%.

EXERCISE CXI.

1. What does "$12\frac{1}{2}$ and 10 off" mean?

2. Give the principles governing Trade Discount.

3. Develop the formulas and commit to memory the relations.

No.	*\$ L*		*p%*	*\$ D*		*\$ S*	
4.	580		10, 10	?		?	
5.	460		?	69		?	
6.	500		20, ?	?		375	
7.	?		15, 10	?		459	
8.	720		? 20	?		432	
9.	?		15	63	81	?	
10.	?		?	304	04	1216	16

11. A man sold goods listed at \$437 at 20% off. What did they sell for?

12. Goods sold at 25 and 5 off of a list price of \$540. Find the amount of the discount.

13. Goods marked at \$380 sold at $\frac{1}{5}$, 10, and 10 off. Find the selling price.

14. Merchandise sold at 20 and 15 off for \$1360. Find the list price.

15. I bought an article, marked it at $20, and sold it at 30 and ( ) off for $12.60. Fill the blank.

16. How must I mark goods that cost me $450 so that I may sell 10 off of list price at a gain of 20%.

17. I sold an article at 20 and 10 off. If the discounts amount to $3.50, for how much did the article sell?

NOTE.—The 1st discount is 20% of the list price, and the 2d discount is 10% of 80% of list price.

135. Commission.—Very often merchandise, especially such as produce and live stock, is bought and sold in distant markets. Usually, such buying and selling are not done by the parties furnishing the money and the commodities, but by agents employed for that purpose. An agent, or commission merchant, charges for his services a certain per cent on the amount he receives in selling or the amount he pays out in buying. This charge is called **Commission.**

PRINCIPLES: *1. The commission is always some number of per cent of the price of that which is bought or sold.*

2. The proceeds are equal to the selling price minus the commission.

3. The amount is equal to the selling price plus the commission.

TERMS.

S, price of the sale or purchase.
p, number of per cent of commission.
C, commission.
P, part left after deducting the commission.
A, amount of investment and commission.

NOTE.—S is selected to represent the price of both *purchases* and *sales*—that on which commission is based.

Development of the formulas for commission.

$$C = \frac{pS}{100}. \text{ (Prin. 1.)}$$

7. An article, listed at \$20, sold at ( ) and 10 off for \$13.50. Fill the blank.

Solution: (1) $13.50 = \dfrac{20(100-p)\times 90}{100 \times 100}$.

$10000 \times (1) =$ (2) $135000 = 1800(100-p)$.

(3) $135000 = 180000 - 1800p$.

Transpose, (4) $1800p = 180000 - 135000$.

(5) $1800p = 45000$.

(6) $p = \dfrac{45000}{1800} = 25$.

Answer, 25%.

EXERCISE CXI.

1. What does "$12\frac{1}{2}$ and 10 off" mean?

2. Give the principles governing Trade Discount.

3. Develop the formulas and commit to memory the relations.

No.	*\$ L*		*p%*	*\$ D*		*\$ S*	
4.	580		10, 10	?		?	
5.	460		?	69		?	
6.	500		20, ?	?		375	
7.	?		15, 10	?		459	
8.	720		? 20	?		432	
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C, commission.
P, part left after deducting the commission.
A, amount of investment and commission.

NOTE.—S is selected to represent the price of both *purchases* and *sales*—that on which commission is based.

Development of the formulas for commission.

$$C=\frac{pS}{100}. \quad \text{(Prin. 1.)}$$

RELATION I. *Abstractly, the commission is equal to the product of the purchase (or sale) and the number of per cent of commission, divided by 100.*

To find the proceeds: (Prin. 2.)

$$P=S-C=S-\frac{pS}{100}=\frac{100S-pS}{100}; \text{ or,}$$

$$P=\frac{S(100-p)}{100}.$$

RELATION II. *Abstractly, the proceeds are equal to the product of the purchase (or sale) and 100 minus the number of per cent of commission, divided by 100.*

To find the amount: (Prin. 3.)

$$A=S+C=S+\frac{pS}{100}; \text{ or,}$$

$$A=\frac{S(100+p)}{100}.$$

RELATION III. *Abstractly, the amount of investment and commission is equal to the product of the purchase (or sale) and 100 plus the number of per cent, divided by 100.*

EXAMPLES.

1. I sell goods for \$1430, commission 4%. Find my commission.

Solution: $C=\frac{1430\times 4}{100}=57.2.$

Answer, \$57.20.

2. How much must an agent remit to his principal, if he sell goods to the amount of \$1200, commission 2% ?

Solution: $P=\frac{1200\times 98}{100}=1176.$

Answer, \$1176.

3. I send my agent \$1442 to be invested in goods, commission 3%. What is the price of the goods bought?

Solution: (1) $A = \dfrac{S\,(100+p)}{100}.$

(2) $1442 = \dfrac{S \times 103}{100}.$

(3) $S = \dfrac{1442 \times 100}{103} = 1400.$

Answer, \$1400.

4. A lawyer charges \$35 for collecting a debt of \$700. Find his % of commission.

Solution: (1) $\dfrac{700p}{100} = 35.$

(2) $p = \dfrac{35 \times 100}{700} = 5.$

Answer, 5%.

EXERCISE CXII.

1. What is commission? A commission merchant?
2. Name and explain the terms used in commission.
3. Develop the formulas for commission.
4. Commit to memory the principles and relations.

No.	\$ *S*		*p* %	\$ *C*		\$ *P*		\$ *A*	
5.	724	50	4	?		?			
6.	840		$2\frac{1}{2}$	?		?			
7.	920		?	?		892	40		
8.	128	80	?	6	44	?			
9.	?		?	48	75	1901	25		
10.	?		3	?		1225	11		
11.	?		2	26	50			?	
12.	?		$1\frac{1}{2}$	?				588	70
13.	?		?	14				854	

14. A commission merchant sells 8000 lb. of bacon at 10¢ per pound; he pays freight \$50, and drayage \$2. If his commission is 2%, what are the net proceeds?

NOTE.—"Net proceeds" are the balance after deducting all expenses.

15. An agent receives \$475 with which to purchase goods. If his commission is $1\frac{1}{2}$%, what amount does he invest?

16. My agent sells 50 head of hogs averaging 225 pounds, at 5¢ per pound. He invests the net proceeds in wheat at \$.50 per bushel. His commission for selling is 3%; for buying, 2%. How much wheat does he buy?

17. My commission at 2% for selling 600 hogs, averaging 225 lb. each, is \$135. What did I get for the hogs? how much per hog? how much per pound?

18. I received an account sales, stating that my net proceeds were \$4301.30; freight, \$55.70; drayage, \$8.00; commission, \$135. Find the rate of commission.

NOTE.—An "*account sales*" is a final statement made by the agent to his principal. It shows (1) the selling price of the goods, (2) all the expenses, including commission, and (3) the proceeds to be remitted.

19. An agent, after deducting his commission, 5%, other expenses, \$42.24, had \$983.76 to remit. Find the amount of sales.

20. Sold flour at $3\frac{1}{2}$% commission; invested $\frac{2}{3}$ of the proceeds in coffee at $1\frac{1}{2}$% commission, and remitted the balance, \$432.50. What was the value of the flour, the coffee, and each commission?

NOTE.—All of the $\frac{2}{3}$ of the sales of the flour was invested in coffee. The commissions are taken out of the remaining $\frac{1}{3}$ of the sales of the flour.

136. Stocks and Bonds.

NOTE.—Children in the public schools usually know little or nothing about Stock Companies or Corporations. A day or two spent in working

out the eight examples below will be of great service to the pupils. Organize the Galena Mining Company, have pupils represent Jones, Smith, Brown, etc., write out some of the certificates of stock, discuss the meaning of dividends, surplus, and so on. Pupils cannot intelligently solve problems which they do not understand.

CERTIFICATE OF STOCK.

No. 1. Capital Stock, $200,000. Shares, $100 each. 200 Shares.

GALENA MINING COMPANY,

GALENA, KANSAS.

This is to certify that William Smith is entitled to Two Hundred Shares in the Capital Stock of the Galena Mining Company. Transferable only on the books of the Company, in person or by attorney, upon surrender of this certificate.

In Witness Whereof, the seal of said Company is

(L. S.) hereto affixed, this 10th day of Feb., 1890.

Attested by Galen Terry,
William Smith, President.
Secretary.

The *Par Value*, or face value, of a share of stock is what it is valued at on the face of the certificate.

The *Market Value* of a share of stock is its selling price in the market.

NOTE.—When a share sells for more than its face value, it is said to be *above par*, or *at a premium*. When a share sells for less than its face value, it is said to be *below par*, or *at a discount*.

Stock companies usually divide their earnings (profits or gains) annually. The amount thus divided is called the *dividend*. Whoever owns a share of stock receives the dividend paid on that share.

Buying or selling of stocks is sometimes done through a *broker* (agent), who charges for his services a *brokerage* (commission) of so many % of the *par value* of the stocks bought or sold.

EXAMPLES.

The Galena Mining Company was organized and incorporated under the laws of Kansas, January 1, 1890, with the following stockholders:

William Jones, 200 shares................	$20000
William Smith, 400 shares..............	40000
James Brown, 350 shares.................	35000
Galen Terry, 400 shares..................	40000
G. W. Jolley, 650 shares..................	65000
Capital stock (C. S.)	$200000

At the close of the year 1890, the net earnings of the company were $22480.50.

1. If the company reserves a surplus in the treasury of $2480.50, what will be the rate of dividend (R. D.)?

NOTE.—After reserving a *surplus* to pay running expenses, what is left of the earnings or gains is divided among the stockholders.

Solution: (1) $22480.50 − $2480.50 = $20000. Amt. to be divided.
(2) $20000 = ( )% of C. S.? (Question.)
(3) $200000 = 100% of C. S. (Basis.)
$\frac{1}{10}$ of (3) = (4) $20000 = 10% of C. S., answer.

2. What is Smith's share of the dividend (his C. S., $40,000)?

Solution: (1) 10% of his C. S. = $( )? (Question.)
(2) 100% of his C. S. = $40000. (Basis.)
$\frac{1}{10}$ of (2) = (3) 10% of his C. S. = $4000, answer.

NOTE.—Brokerage is the commission an agent (broker) charges for buying or selling stocks or bonds. *But unlike commission, brokerage is always some per cent of the* PAR VALUE *of the stocks or bonds, and not some per cent of the cost.*

3. Jones sells some of his shares to W. W. Miller at 90 (10% below par), brokerage $\frac{1}{8}$ %, for $13518.75. Find the face value of the stocks sold.

Solution : (1) 90% of par value = price of stocks.
(2) $\frac{1}{8}$% of par value = brokerage.
(3) 100% of par value = \$( ) ? (Question.)
(4) 90$\frac{1}{8}$% of par value = \$13518.75. (Basis.)
$\frac{8}{721}$ of (4) = (5) 1% of par value = \$150.
100 × (5) = (6) 100% of par value = \$15000, answer.

At the beginning of the year 1891, Terry sells his shares to J. O. Bowers; Jolley sells 300 shares to G. W. Greene at 120 (20% above par); at the end of the year 1891, the company finds that, after reserving a surplus of \$3124.40, it can declare a dividend of 6%.

NOTE.—For the sake of brevity the "*question*" will be omitted in the remainder of these solutions.

4. Find the net earnings of the company for 1891.

Solution : (1) 100% of C. S. = \$200000. (Basis.)
$\frac{1}{100}$ of (1) = (2) 1% of C. S. = \$2000.
6 × (2) = (3) 6% of C. S. = \$12000.
(4) \$12000 + 3124.40. = \$15124.40, answer.

5. What did Greene's 300 shares cost him (at 120)?

Solution : (1) 100% of 300 shares = \$30000. (Basis.)
$\frac{1}{100}$ of (1) = (2) 1% of 300 shares = \$300.
120 × (2) = (3) 120% of 300 shares = \$36000, answer.

6. What was Greene's dividend?

Solution : (1) 100% of 300 shares = \$30000. (Basis.)
$\frac{1}{100}$ of (1) = (2) 1% of 300 shares = \$300.
6 × (2) = (3) 6% of 300 shares = \$1800, answer.

7. What is Greene's rate of income on his investment?

NOTE.—The "*rate of income*" is always some number of per cent of the *investment* or cost.

Solution : (1) \$36000 = 100% of his investment. (Basis.)
$\frac{1}{20}$ of (1) = (2) \$1800 = 5% of his investment, answer.

8. If Bowers's rate of income is 8% for 1891, for how much below par did he buy his stock? (R. D. was 6%).

Solution: (1) 100% of investment = ( )% of his C. S.?
(2) 8% of investment = 6% of C. S.
$\frac{1}{8}$ of (2) = (3) 1% of investment = $\frac{3}{4}$% of C. S.
100 × (3) = (4) 100% of investment = 75% of C. S., or 25% below par, answer.

Bonds are promissory notes issued by a public corporation (nation, state, county, city), or a private corporation (railroad company, banking company, mining company, etc.). Bonds have a *face value* (*par value*), *a rate of interest*, and when bought and sold they have a *market value*. *Brokerage* may be charged by an agent for buying or selling bonds.

EXAMPLES.

1. I own 6's to the amount of $5500; what is my annual income, or interest?

NOTE.—"6's" means that the rate of interest is 6%.

Solution: (1) 100% of P. V. = $5500. (Basis.)
$\frac{1}{100}$ of (1) = (2) 1% of P. V. = $55.
6 × (2) = (3) 6% of P. V. = $330, answer.

2. A buys bonds amounting to $50000 at 104, brokerage $\frac{1}{4}$%; what do they cost him?

Solution: (1) 100% of P. V. = $50000. (Basis.)
$\frac{1}{100}$ of (1) = (2) 1% of P. V. = $500.
$104\frac{1}{4}$ × (2) = (3) $104\frac{1}{4}$% of P. V. = $52125, answer.

3. A invests $7210 in bonds at 90, brokerage $\frac{1}{8}$%; what is the face value of the bonds?

Solution: (1) $90\frac{1}{8}$% of P. V. = $7210. (Basis.)
$\frac{8}{721}$ of (1) = (2) 1% of P. V. = $80.
100 × (2) = (3) 100% of P. V. = $8000, answer.

4. B invests \$24000 in 4's, at 80; what is his interest?

Solution: (1) 80% of P. V. = \$24000. (Basis.)
$\frac{1}{80}$ of (1) = (2) 1% of P. V. = \$300.
4 × (2) = (3) 4% of P. V. = \$1200, answer.

5. How must C buy 6's to yield an income of 5%?

Solution: (1) 5% of M. V. = 6% of P. V. (Basis.)
20 × (1) = (2) 100% of M. V. = 120% of P. V., answer.

6. Which furnishes the better income on the investment—5's at 75, or 6's at 80?

In the First: (1) 75% of P. V. = 100% of investment. (Basis.)
$\frac{1}{15}$ of (1) = (2) 5% of P. V. = $6\frac{2}{3}$% of investment.

In the Second: (1) 80% of P. V. = 100% of investment. (Basis.)
$\frac{1}{80}$ of (1) = (2) 1% of P. V. = $\frac{5}{4}$% of investment.
6 × (2) = (3) 6% of P. V. = $7\frac{1}{2}$% of investment.
(4) $7\frac{1}{2}$% of investment − $6\frac{2}{3}$% of investment = $\frac{5}{6}$% of investment, in favor of the second, answer.

The problems of stocks and bonds are so nearly alike that one set of principles and formulas applies to both.

PRINCIPLES: *1. The discount, premium, dividend (or interest) and brokerage are each some number of per cent of the par value.*

2. The per cent (or rate) of income is some number of per cent of the cost.

3. The market value is equal to the par value plus the premium, or minus the discount.

4. The cost is equal to the market value plus the brokerage.

TERMS.

P, par value.
M, market value.
C, cost.
B, brokerage.
D, dividend (stocks).
I, interest (bonds).

In., income.
Pr., premium.
Dis., discount.
R, number of % of the dividend or interest.
r, number of % of the income on the investment.
p, number of % of premium or discount.
p', number of % of the brokerage.

Develop the formulas for Stocks and Bonds:

1. $Pr.$ or $Dis. = \frac{pP}{100}$. (Prin. 1.)

2. $B = \frac{p'P}{100}$. (Prin. 1.)

3. D or $I = \frac{rP}{100}$. (Prin. 1.)

Since the income on the investment is the same as the dividend (stocks) or the interest (bonds), it follows that—

4. D or $I = \frac{rC}{100}$. (Prin. 2.)

When there is no brokerage,

5. D or $I = \frac{rM}{100}$. (Why?)

By comparing formulas 3 and 4, we obtain

$$\frac{RP}{100} = \frac{rC}{100}; \text{ or,}$$

6. $RP = rC$.

When there is no brokerage,

7. $RP = rM$.

When stocks or bonds are at a premium,

(1) $M = P + Pr.$ (Prin. 3.)
(2) $C = P + Pr + B.$ (Prin. 4.)
(1) = (3) $M = P + \frac{pP}{100} = \frac{100\,P + pP}{100}$; or,

$$8.\quad M=\frac{P(100+p)}{100}.$$

$$(2)=(4)\quad C=P+\frac{pP}{100}+\frac{p'P}{100}=\frac{100\,P+pP+p'P}{100}\text{; or,}$$

$$9.\quad C=\frac{P(100+p+p')}{100}.$$

When stocks or bonds are at a discount, show that—

$$10.\quad M=\frac{P(100-p)}{100}\text{, and}$$

$$11.\quad C=\frac{P(100-p+p')}{100}.$$

EXAMPLES.

NOTE.—In this book, the par value of a share or a bond is \$100, unless it should be otherwise stated.

1. Find the discount on 20 shares of stock sold at 20% below par.

Solution: $Dis.=\frac{20\times2000}{100}=400.$

Answer, \$400.

2. Find the premium on 60 shares of stock sold at $12\frac{1}{2}\%$ premium.

Solution: $Pr.=\frac{12\frac{1}{2}\times6000}{100}=750.$

Answer, \$750.

3. Find the brokerage at $\frac{1}{8}\%$ on 40 shares of stock.

Solution: $B=\frac{\frac{1}{8}\times4000}{100}=5.$

Answer, \$5.

4. Find the dividend on 50 shares of 8% stock.

Solution: $D=\frac{8\times5000}{100}=400.$

Answer, \$400.

5. Find the rate of interest on 50 bonds that yield $450 annual interest.

Solution: (1) $I=\frac{RP}{100}$.

(2) $450=\frac{5000\,R}{100}$.

(3) $R=\frac{450\times 100}{5000}=9$.

Answer, 9%.

6. If the annual interest of $920 on bonds pays an income of 8%, find the cost.

Solution: (1) $I=\frac{rC}{100}$.

(2) $920=\frac{8\,C}{100}$.

(3) $C=\frac{920\times 100}{8}=11500$.

Answer, $11500.

7. Find the cost of 45 shares of stock bought at 110.

Solution: $M=\frac{4500\times 110}{100}=4950$.

Answer, $4950.

NOTE.—The "110" $=(100+p)=(100+10)$.

8. Find the cost of 15 bonds at 60, brokerage $\frac{1}{8}$.

Solution: $C=\frac{1500\times 60\frac{1}{8}}{100}=901.875$.

Answer, $901.875.

NOTE.—The $60\frac{1}{8}=(100-p+p')=100-40+\frac{1}{8})$.

9. I buy 6% bonds at 80; find my rate of income.

NOTE.—Assuming 1 bond as a basis, the P is $100, the M is $80.

Solution: (1) $RP=rM$.

(2) $6\times 100=80r$.

(3) $r=\frac{6\times 100}{80}=7\frac{1}{2}$.

Answer, $7\frac{1}{2}$%.

10. I bought 5's, $\frac{1}{4}$ brokerage, for \$903.75. If they pay me $8\frac{72}{241}\%$ on my investment, find the par value, the % of discount, the brokerage, and the market value.

NOTE.—"5's" means bonds that bear 5% interest per annum.

Solution: (1) $RP = rC$.

(2) $5P = 8\frac{72}{241} \times 903.75$.

(3) $P = \dfrac{2000 \times 903.75}{241 \times 5} = 1500.$

Par value, \$1500.

(4) $B = \dfrac{\frac{1}{4} \times 1500}{100} = 3.75.$

Brokerage, \$3.75.

(5) $M = \$903.75 - \$3.75 = \$900.$

(6) Dis. $= \$1500 - \$900 = \$600.$

(7) $\dfrac{1500p}{100} = 600.$

(8) $p = \dfrac{600 \times 100}{1500} = 40.$

Discount, 40%.

EXERCISE CXIII.

1. Discuss the organization of a stock company — the capital stock, the issuing of shares, and the distribution of earnings.

2. Define brokerage, par value, market value, premium, discount.

3. When a company loses instead of gaining during the year, how is the assessment made to pay this loss?

4. What is a bond? The interest?

5. Commit to memory the principles.

6. Develop the formulas, and write out the relation expressed by each formula.

7. Find the discount on 34 shares of stock, sold in market at 30% below par.

8. Find the premium on 75 shares of 10% stock, sold in market at 30% premium.

9. I bought bonds at 105 for \$7350. Find the par value.

10. 40 shares of stocks sold in market for \$3600. Find the per cent. of discount.

11. I bought 30 shares of 6% stock. Find my annual dividend.

12. My interest on 6% bonds is \$540. Find the face value of the bonds.

13. I receive a dividend of \$400, which is $6\frac{2}{3}$% on my investment. Find the cost of the bonds.

14. I bought 60 shares of 8% stock for 110. Find the rate of income.

15. I bought 5% stock at 70. Find the rate of income.

16. I bought 6% bonds so as to pay me 10% on my investment. Find the market price.

17. Find the cost of 40 shares of stock, sold at 80, brokerage $\frac{1}{4}$%.

18. Find the cost of 75 shares of stock, sold at 120, brokerage $\frac{1}{8}$%.

19. How many shares of stock at $77\frac{7}{8}$, brokerage $\frac{1}{8}$%, can be bought for \$15600?

20. Which pays the better income on the investment, and how much — 7's at 90, or 6's at 75?

21. A man owns 5% railroad stocks worth 90 and 4% bonds worth 80. If his income from each is \$720, find the par value of each.

22. In No. 21, find the rate of income on each investment.

23. How much money must be invested in 4's at 90 to yield an annual income of \$120?

24. On bonds bought at 70, my rate of income is 5%. This income annually amounts to \$1750. Find the amount of the bonds and the rate of interest.

137. Taxes.—A *tax* is a sum of money levied on persons (poll tax) or property (property tax).

PRINCIPLES: *1. A poll tax is a certain sum required usually of those citizens who have the right to vote.*

2. A property tax is some number of per cent of the value of the property taxed.

TERMS.

V, value of property.
p, number of % of the taxes.
T, taxes.

Formula: $T=\frac{pV}{100}$. (Prin. 2.)

EXAMPLES.

1. In a certain county there were 7570 polls, at $1.25. Find the poll tax.

Solution: $T=7570\times\$1.25=\$9462.50$, answer.

2. The property of the county was valued at $5897500. Find the tax raised by a $\frac{1}{2}$% (5 mills) levy.

Solution: $T=\frac{\frac{1}{2}\times 5897500}{100}=29487.50$.
Answer, $29487.50.

NOTE.—In tax levies is almost the only place in business where the denomination *mill* is used. 1 mill $=\frac{1}{10}$¢. A 5-mill levy is a levy of 5 mills or $\frac{1}{2}$¢ on $1; or, $\frac{1}{2}$% of the value of the property taxed.

EXERCISE CXIV.

1. Define poll tax; property tax.

2. Give the principle governing poll tax; property tax.

3. Give the relation expressed by the formula above.

4. If the number of polls be represented by N, and the tax on each poll by t, give the formula for finding poll tax, T.

5. Find the tax on 1200 polls, at $1.35 each.

6. Find the property tax on a farm valued at $10400, if the levy is 14 mills ($1\frac{2}{5}\%$).

7. What is the valuation of A's property who pays $51 on a 15 mill ($1\frac{1}{2}\%$) levy?

8. A school district has $20000 taxable property, and desires to raise $240 teachers' wages. What % levy must they run?

9. A tax of $787.50 is raised on 25 polls at $1.50 each, and a property tax of 6 mills. Find the value of the property taxed.

10. Find the % of the tax levy, when $17500 worth of property, and 40 polls at $1 each, produce a tax of $285.

138. Duties.—Taxes levied by the government upon imported merchandise are called **Duties.** Duties are of two kinds, *specific* duties and *ad valorem* duties.

A *Specific duty* is levied upon the amount of the merchandise imported. An *Ad valorem duty* is levied upon the value of the merchandise imported.

Tare, *leakage*, and *breakage* are allowances deducted from the gross amount of imported merchandise. These deductions are made *before* levying duties.

Tare is the deduction made for the weight of the boxes, barrels, etc., containing the merchandise.

Leakage is the deduction made for the loss of liquids shipped in barrels or casks. The amount to be deducted is determined by *gauging*.

Breakage is the deduction made for the loss of liquids shipped in bottles.

PRINCIPLES: *1. Specific duty is a certain sum per pound, yard, gallon, etc., on the merchandise imported.*

2. Ad valorem duty is some number of per cent of the net value of the merchandise imported.

TERMS.

V, net value of merchandise.
p, number of % of ad valorem duty.
D, the duty.

Formula for ad valorem duty:

$$D=\frac{pV}{100}. \quad \text{(Prin. 2.)}$$

EXAMPLES.

1. What is the duty on 560 bottles of wine, duty 60¢ per dozen, breakage $12\frac{1}{2}\%$?

Solution: (1) Breakage $= 12\frac{1}{2}\%$ of 560 bottles $=$ 70 bottles.
(2) 560 bottles $-$ 70 bottles $=$ 490 bottles $= 40\frac{5}{6}$ dozen bottles.
(3) *Duty* $= 40\frac{5}{6} \times 60¢ = \$24.50$, answer.

2. Find the duty on 400 bags of coffee, each weighing 180 lb. gross, invoiced at 8¢ per lb., tare 5%, ad valorem duty 20%.

Solution: (1) Amount $= 400 \times 180$ lb. $=$ 72000 lb.
(2) *Tare* $=$ 5% of 72000 lb. $=$ 3600 lb.
(3) 72000 lb. $-$ 3600 lb. $=$ 68400 lb.
(4) *Value* $= 68400 \times 8¢ = \$5472$.
(5) $D=\frac{20 \times 5472}{100}=1094.40.$
Answer, \$1094.40.

3. A merchant imports 10 casks of brandy, each containing 30 gallons, worth \$1.40 per gallon; duties are 10¢ per gallon, and 25% ad valorem; leakage, 10%. Find the whole duty.

Solution: (1) *Amount* $= 10 \times 30$ gal. $=$ 300 gal.
(2) *Leakage* $=$ 10% of 300 gal. $=$ 30 gal.
(3) 300 gal. $-$ 30 gal. $=$ 270 gal.
(4) Value $= 270 \times \$1.40 = \$378$.
(5) *Sp. D.* $= 270 \times 10¢ = \$27$.
(6) *Ad. D.* $=$ 25% of \$378 $=$ \$94.50.
(7) \$27 + \$94.50 $=$ \$121.50, answer.

EXERCISE CXV.

1. Define duty, specific duty, ad valorem duty, tare, leakage, breakage.

2. Commit the principles.

3. Give the relation expressed by the formula.

4. Write the formula for specific duty (D), when the amount (A) of the merchandise and the duty (d) on the unit of amount is given.

5. Find the duty on 500 yd. of silk invoiced at \$1.50, at 20% ad valorem.

6. What is the duty at 25% ad valorem on 400 boxes of raisins of 40 lb. each, invoiced at 9¢ per lb., tare 10% ?

7. The ad valorem duty on 1500 yd. Irish linen, valued at 40¢ per yd., is \$210. Find the % of duty.

8. Find the specific duty on 750 yd. broadcloth at 50¢ per yd.

9. What is the specific duty per gallon, when a duty of \$540 is paid on 50 casks of wine of 30 gallons each, leakage 10% ?

10. Find the whole duty on 4500 lbs. of wool, invoiced at 20¢ per lb., specific duty 10¢ per lb., ad valorem duty 12%, tare 8%.

139. Insurance.—Insurance is indemnity against loss by damage or death. Insurance is of two kinds, *property* insurance and *personal* insurance.

Property insurance is indemnity for loss of property, caused by fire, lightning, tornadoes, death of stock, and the like.

Personal insurance is indemnity for loss caused by death, sickness, or accident.

A *Policy* is the written contract between the insurance company and the person insured.

A policy which insures the payment of a certain sum only at the death of the person insured, is called a *Life policy*. A policy which insures the payment of a certain sum at a specified time, or at death occurring before that time, is called an *Endowment policy*.

Premium is the sum paid for insurance.

The *Risk* is the amount insured.

NOTE.—Insurance companies seldom agree to insure property for its full value. They usually make the risk $\frac{2}{3}$ or $\frac{3}{4}$ of the full value.

PRINCIPLES: *1. In property insurance, the premium, due annually, is some number of per cent of the risk.*

2. In life insurance, the premium is usually a certain sum per \$1000, payable annually (semi-annually or quarterly).

TERMS.

R, the risk.
p, number of % of the premium.
P, the premium.

Formula for Property Insurance:

$$P=\frac{pR}{100}.\quad \text{(Prin. 1.)}$$

EXAMPLES.

1. Find the premium at 2% on a risk of \$8400.

Solution: $P=\frac{2\times 8400}{100}=168.$

Answer, \$168.

2. The annual premium is \$480, paid on a factory valued at \$48000, insured at $\frac{2}{3}$ of its value. Find the rate of insurance.

Solution: (1) $\frac{2}{3}$ of \$48000 = \$32000, risk.

(2) $480=\frac{p\times 32000}{100}$. (Why?)

(3) $p=\frac{480\times 100}{32000}=1\frac{1}{2}$.

Answer, $1\frac{1}{2}$%.

3. A company insured a vessel worth \$60000 for $\frac{2}{3}$ its value at 2%; they reinsured $\frac{3}{4}$ of their risk at $1\frac{3}{4}$%. Find the premium retained by the first company.

Solution: (1) $\frac{2}{3}$ of \$60000 = \$40000, whole risk.
(2) $\frac{3}{4}$ of \$40000 = \$30000, second risk.
(3) *1st premium* = 2% of \$40000 = \$800.
(4) *2d premium* = $1\frac{3}{4}$% of \$30000 = \$525.
(5) \$800 − \$525 = \$275, answer.

4. What will be the annual premium on a life policy of $4000 at $25.40 per $1000 ?

Solution : *Premium* = 4 × $25.40 = $101.60, **answer.**

5. If Mr. Jones was insured for $5000, at an annual premium of $36.40 per $1000, what amount did the company pay out more than it received, if Mr. Jones died after making 11 annual payments ?

Solution : (1) *Premium* = 11 × 5 × $36.40 = $2002. (Why ?)
(2) $5000 − $2002 = $2998, **answer.**

EXERCISE CXVI.

1. Define insurance, policy, premium.

2. Define a life policy, endowment policy.

3. Give the principles.

4. Give the relation expressed by the formula.

5. How do you find the premium on a life insurance policy when the amount of the policy and the premium per $1000 is given ?

6. Find the premium at 3% on a risk of $6500.

7. What is the rate of insurance, if the premium on $5420 is $135.50 ?

8. The premium at $2\frac{3}{4}$% on a certain risk was $99. Find the risk.

9. A vessel valued at $50400 was insured for $\frac{3}{4}$ its value at $1\frac{1}{2}$%. The first company re-insured $\frac{2}{3}$ of the risk in a second company, paying $441 premium. Find the first premium, and the second rate of insurance.

10. Find the premium paid in 15 yr. on a life policy of $6000, at $27.50 per $1000 annually.

11. A man took out an endowment policy for $4000 and died after making 10 annual payments. If his estate received $2682 more than he had paid out, what was the annual premium per thousand ?

3. PERCENTAGE WITH TIME.

140. Terms used in Interest. Interest is a sum paid for the use of money. The amount of money used is the **Principal.** The per cent of the principal which equals one year's interest is the **Rate.** If only the principal bear interest, the interest is called **Simple Interest.**

When by the terms of a note interest is payable annually, (semi-annually, or quarterly,) and is not paid when due, this *annual interest* (interest due annually, semi-annually, or quarterly) bears interest at the stipulated or legal rate until paid. This is called **Annual Interest.** A note which reads, "and if the interest be not paid annually [semi-annually, or quarterly], it becomes as principal, and bears the same rate of interest," bears **Compound Interest.**

141. Time. Since the rate in interest problems is "*per annum*," or by the year, the time should be expressed in years. By the common method of counting time in interest problems,

$$1 \text{ mo.} = \tfrac{1}{12} \text{ yr.}$$
$$1 \text{ da.} = \tfrac{1}{360} \text{ yr.}$$

This method of counting time is called the *common interest method.*

Example: Express in terms of years (1) 3 mo., (2) 17 da., (3) 6 mo. 8 da., (4) 5 mo. 18 da., (5) 2 yr. 4 mo. 12 da.

Results: (1) 3 mo. $= \frac{3}{12}$ yr.
(2) 17 da. $= \frac{17}{360}$ yr.
(3) 6 mo. 8 da. $= \frac{188}{360}$ yr.
(4) 5 mo. 18 da. $= \frac{5.6}{12}$ yr.
(5) 2 yr. 4 mo. 12 da. $= \frac{28.4}{12}$ yr.

NOTE.—In using these expressions in the interest problems to follow, cancellation will be used after the whole problem is stated; therefore

these expressions need *not* be reduced to lowest terms. When the number of days is a multiple of 3, divide by 3 and express the result as tenths of a month; as in (4) and (5) above. Thus:

$$15 \text{ da.} = \frac{.5}{12} \text{ yr.}$$

$$33 \text{ da.} = \frac{1.1}{12} \text{ yr.}$$

$$24 \text{ da.} = \frac{.8}{12} \text{ yr., and so on.}$$

EXERCISE CXVII.

Express in years:

1. 6 mo.
2. 8 mo.
3. 9 mo.
4. 5 mo.
5. 10 mo.
6. 30 da.
7. 12 da.
8. 24 da.
9. 2 yr. 4 mo.
10. 1 yr. 8 mo.
11. 5 mo. 15 da.
12. 6 mo. 24 da.
13. 7 mo. 21 da.
14. 1 yr. 6 mo. 12 da.
15. 2 yr. 4 mo. 10 da.
16. 1 yr. 11 mo. 23 da.

Find the time in years from —

17. Mar. 20, 1900, to Sept. 25, 1901.

	yr.	*mo.*	*da.*
Form:	1901	9	25
	1900	3	20
	1	6	$5 = \frac{545}{360}$ yr.

NOTE.—Subtract as in compound numbers (p. 176).

18. Sept. 10, 1890, to Jan. 20, 1896.
19. June 1, 1897, to Jan. 5, 1898.
20. July 10, 1895, to June 5, 1896.
21. Feb. 1, 1899, to Nov. 1, 1900.
22. Dec. 25, 1884, to July 5, 1889.
23. Nov. 21, 1893, to Oct. 15, 1895.
24. Aug. 30, 1898, to April 10, 1900.

In *Bankers' Interest* (bank discount), the exact number of days is counted; but, in expressing the time in years, 360 da. is considered 1 yr.

Example: Find the time in years from Jan. 4, 1900, to Feb. 15, 1900 (bankers' interest).

Process: (1) From Jan. 4, 1900, to Feb. 15, 1900 = 42 da.
(2) 42 da. = $\frac{42}{360}$ yr.

NOTE.—After Jan. 4, there are 27 da. in Jan. and 15 da. in Feb.; or, 42 da. in all.

In *Exact Interest*, the exact number of days is counted; but, in expressing the time in years, 365 days is considered 1 yr. (366 days in leap year).

Example: Find the time in years from July 9, 1897, to Oct. 12, 1897 (exact interest).

Process: (1) From July 9, to Oct. 12 = 95 days.
(2) 95 da. = $\frac{95}{365}$ yr.

EXERCISE CXVIII.

Find the time in years (bankers' interest) from—

1. Feb. 12, 1900, to July 1, 1900.
2. Feb. 4, 1899, to April 10, 1899.
3. Mar. 17, 1896, to June 25, 1896.
4. Jan. 19, 1896, to May 10, 1896.
5. July 29, 1895, to Oct. 11, 1895.
6. Dec. 10, 1898, to Jan. 27, 1899.

Find the time in years (exact interest) from—

7. Nov. 12, 1893, to Jan. 1, 1894.
8. Oct. 21, 1900, to Dec. 21, 1900.
9. May 5, 1896, to Aug. 1, 1896.
10. Sept. 28, 1892, to Dec. 1, 1892.
11. Jan. 12, 1896, to Mar. 1, 1896.
12. Dec. 20, 1897, to Feb. 15, 1898.

142. Simple Interest. (See definition, p. 281.)

TERMS.

P, the principal.
T, the time.
R, the number of % of the rate.
I, the interest.
A, the amount of principal and interest.

Develop the formulas for simple interest.

We know—

(1) Int. on \$1 for 1 yr. at 1% = \$$\frac{1}{100}$.

(2) Int. on \$$P$ for 1 yr. at 1% = \$$\frac{P}{100}$.

(3) Int. on \$$P$ for T yr. at 1% = $\frac{P \times T}{100}$.

(4) Int. on \$$P$ for T yr. at R% = $\frac{P \times T \times R}{100}$.

Now, expressing (4) abstractly, we have the interest formula—

$$1.\ I = \frac{PRT}{100}.$$

RELATION I. *Abstractly, the interest is equal to the continued product of the principal, rate, and time, divided by 100.*

We know—

$$(1)\ A = P + I = P + \frac{PRT}{100}.$$

$$(1) = (2)\ A = \frac{100\,P + PRT}{100};\ \text{or,}$$

$$2.\ A = \frac{P(100 + RT)}{100}.$$

RELATION II. *Abstractly, the amount of a given principal bearing interest at a given rate for a given time is equal to the product*

of the principal and 100 plus the product of the number of per cent of the interest and the time, divided by 100.

Note.—Remember that in using these formulas the time must be expressed in years before applying the formula.

EXAMPLES.

1. Find the interest on $300 for 2 yr. 6 mo. at 8%.

Solution : (1) 2 yr. 6 mo. $= 2\frac{1}{2}$ yr. $= \frac{5}{2}$ yr.

$$(2)\ I = \frac{300 \times 8 \times 5}{100 \times 2} = 60.$$

Answer, $60.

2. Find the interest on $1000 at 8% for 1 yr. 3 mo. 20 da.

Solution : (1) 1 yr. 3 mo. 20 da $= \frac{470}{360}$ yr.

$$(2)\ I = \frac{1000 \times 8 \times 470}{100 \times 360} = 104.44\tfrac{4}{9}.$$

Answer, 104.44\frac{4}{9}$.

3. At what rate will $300 gain $36 in 1 yr. 4 mo.?

Solution : (1) 1 yr. 4 mo. $= \frac{4}{3}$ yr.

$$(2)\ I = \frac{PRT}{100}.$$

$$(3)\ \$36 = \frac{300 \times 4 \times R}{3 \times 100}.$$

$$(4)\ R = \frac{36 \times 3 \times 100}{300 \times 4} = 9.$$

Answer, 9%.

4. In what time will $400 gain $37.60, interest at 8%?

Solution : $$(1)\ 37.6 = \frac{400 \times 8 \times T}{100}.$$

$$(2)\ T = \frac{37.6 \times 100}{400 \times 8} = 1.175.$$

Answer, 1.175 yr. = 1 yr. 2 mo. 3 da.

5. What principal will gain $39 in 1 yr. 3 mo. at 6%?

Solution : (1) 1 yr. 3 mo. $= \frac{5}{4}$ yr.

$$(2)\ 39 = \frac{6 \times 5 \times P}{4 \times 100}.$$

$$(3)\ P = \frac{39 \times 4 \times 100}{6 \times 5} = 520.$$

Answer, $520.

6. Find the amount of \$640 at 9% interest for 1 yr. 5 mo. 15 da.

Solution: (1) 1 yr. 5 mo. 15 da. $=\frac{17.5}{12}$ yr.

(2) $I=\frac{640\times 9\times 17.5}{100\times 12}=84.$

Interest, \$84.

(3) Amount = \$640 + \$84 = \$724, answer.

7. The amount of a note at interest for 1 yr. 4 mo. at 6% is \$432. Find the principal.

Solution: (1) 1 yr. 4 mo. $=\frac{4}{3}$ yr.

(2) $432=\frac{P(100+6\times\frac{4}{3})}{100}=\frac{P\times 108}{100}.$

(3) $P=\frac{432\times 100}{108}=400.$

Answer, \$400.

8. At what rate will any principal double itself in 8 yr.?

NOTE.—Any principal has doubled itself when the interest equals the principal.

Solution: (1) *Interest* $=P.$

(2) $P=\frac{8\,PR}{100}.$ (Why?)

(3) $R=\frac{P\times 100}{8\times P}=12\frac{1}{2}.$

Answer, $12\frac{1}{2}\%$.

NOTE.—In (3), the *P's* cancel.

Another solution: (1) Int. for 8 yr. = 100% of prin. (Why?)

$\frac{1}{8}$ of (1) = (2) Int. for 1 yr. = $12\frac{1}{2}\%$ of prin., answer.

9. In what time will any principal treble itself at 6%?

Solution: (1) *Interest* $=2\,P.$ (Why?)

(2) $2\,P=\frac{P\times 6\times T}{100}.$ (Why?)

(3) $T=\frac{2\times P\times 100}{P\times 6}=33\frac{1}{3}.$

Answer, $33\frac{1}{3}$ yr. = 33 yr. 4 mo.

Another solution: (1) 6% of prin. = int. for 1 yr.
$\frac{1}{6}$ of (1) = (2) 1% of prin. = int. for $\frac{1}{6}$ yr.
200 × (2) = (3) 200% of prin. = int. for $33\frac{1}{3}$ yr., answer.

EXERCISE CXIX.

1. Define *interest*, *principal*, *rate*, *amount*.
2. What is *simple interest?*
3. How does *annual interest* differ from simple interest?
4. How does *compound interest* differ from annual interest?
5. Why should the time be expressed in years?
6. Explain the *common interest* method of counting time.
7. Explain the *bankers' interest* method.
8. Explain the *exact interest* method.
9. Develop the formulas for simple interest.
10. Commit to memory the relations.

Find the simple interest on —

11. \$240 for 2 yr. at 6%.
12. \$78 for 3 yr. at 8%.
13. \$654 for 5 yr. at 5%.
14. \$356 for 6 yr. at $4\frac{1}{2}$%.
15. \$2400 for 4 yr. at 8%.
16. \$468.50 for 3 yr. at 10%.
17. \$763.65 for 2 yr. at 7%.
18. \$95000 for 12 yr. at $5\frac{1}{2}$%.
19. \$86 for 2 mo. at 10%.
20. \$1500 for 3 mo. at 8%.
21. \$684 for 6 mo. at 6%.
22. \$2450.30 for 4 mo. at 7%.
23. \$869 for 9 mo. at 10%.
24. \$1250 for 5 mo. at 12%.
25. \$763.40 for 8 mo. at 9%.
26. \$1386.60 for 10 mo. at 8%.
27. \$79 for 30 da. at 12%.

28. \$156 for 45 da. at 10%.
29. \$288 for 50 da. at 10%.
30. \$855 for 33 da. at 8%.
31. \$926.42 for 15 da. at 12%.
32. \$1346.29 for 60 da. at 6%.
33. \$2400 for 93 da. at 8%.
34. \$1198.75 for 100 da. at 7%.
35. \$248 for 1 yr. 3 mo. at 8%.
36. \$192 for 2 yr. 2 mo. at 7%.
37. \$486 for 4 yr. 6 mo. at 6 %.
38. \$1573.59 for 3 yr. 4 mo. at 8%.
39. \$2576.42 for 5 yr. 6 mo. at 5%.
40. \$986.54 for 6 yr. 8 mo. at 9%.
41. \$3546 for 10 yr. 9 mo. at 4%.
42. \$5684.76 for 15 yr. 8 mo. at $3\frac{1}{2}$%.
43. \$384 for 3 mo. 15 da. at 6%.
44. \$576 for 2 mo. 18 da. at 7%.
45. \$945 for 4 mo. 27 da. at 5%.
46. \$128.73 for 3 mo. 20 da. at 6%.
47. \$894.65 for 6 mo. 12 da. at 8%.
48. \$1246.68 for 6 mo. 15 da. at 8%.
49. \$967.95 for 8 mo. 24 da. at 7%.
50. \$684.88 for 2 mo. 5 da. at 10%.
51. \$129 for 1 yr. 3 mo. 15 da. at 5%.
52. \$2700 for 1 yr. 4 mo. 18 da. at 6%.

Find the amount of—

53. \$678 for 3 yr. 3 mo. 24 da. at 6%.
54. \$940 for 4 yr. 3 mo. 21 da. at 7%.
55. \$1568 for 5 yr. 8 mo. 18 da. at 8%.
56. \$2568.72 for 2 yr. 9 mo. 27 da. at 5 %.
57. \$5468.46 for 7 yr. 10 mo. 19 da. at 4%.
58. \$4698.58 for 6 yr. 7 mo. 25 da. at $5\frac{1}{2}$%.

Find the rate required for—

59. \$800 to produce \$114 interest in 1 yr. 7 mo.
60. \$540 to produce \$64.80 interest in 2 yr.
61. \$276 to produce 11.96 interest in 6 mo. 15 da.
62. \$1500 to produce \$117.50 interest in 1 yr. 6 mo. 24 da.
63. \$450 to produce \$12.60 interest in 8 mo. 12 da.
64. \$780 to produce \$142.35 interest in 1 yr. 9 mo. 27 da.

Find the time required for—

65. \$300 to produce \$52.50 interest at 5%.
66. \$540 to produce \$86.40 interest at 6%.
67. \$1000 to produce \$110 interest at 8%.
68. \$240 to produce \$16.80 interest at 10%.
69. \$5000 to produce \$262.50 interest at 7%.
70. \$840 to produce \$11.76 interest at 8%.
71. \$235 to produce \$2.585 interest at 12%.
72. \$340 to produce \$5.10 interest at 6%.

Find the principal required to produce—

73. \$30 interest in 2 yr. 6 mo. at 6%.
74. \$50.40 interest in 1 yr. 9 mo. at 8%.
75. \$208 interest in 4 yr. 4 mo. at 10%.
76. \$96.35 interest in 1 yr. 8 mo. 15 da. at 6%.
77. \$35 interest in 3 mo. at 7%.
78. \$28.12½ interest in 4 mo. 15 da. at 5%.
79. \$6.40 interest in 45 da. at 8%.
80. \$13.50 interest in 90 da. at 6%.

Find the principal which amounts to—

81. \$570 in 2 yr. 4 mo. at 6%.
82. \$751.20 in 6 mo. 15 da. at 8%.
83. \$904.44$\frac{4}{9}$ in 1 yr. 3 mo. 20 da. at 10%.
84. \$2516.66$\frac{2}{3}$ in 30 da. at 8%.

Find the interest (common interest method) on—

85. $350 from Jan. 10, 1875, to June 15, 1876, at 6%.

86. $500 from May 1, 1880, to Dec. 1, 1884, at 8%.

87. $490 from March 15, 1897, to Jan. 1, 1901, at 8%.

88. $375 from July 5, 1896, to May 30, 1899, at 10%.

89. $400 from June 20, 1895, to Feb. 1, 1898, at 6%.

Find the interest (bankers' interest method) on—

90. $80 from Jan. 8, 1887, to Feb. 20, 1887, at 10%.

91. $250 from July 15, 1890, to Sept. 20, 1890, at 8%.

92. $385 from Aug. 12, 1891, to Nov. 1, 1891, at 12%.

93. $1200 from Jan. 18, 1892, to April 1, 1892, at 8%.

94. $840 from May 12, 1893, to Sept. 5, 1893, at 8%.

Find the exact interest on—

95. $125 from Feb. 1, 1891, to Dec. 1, 1891, at 6%.

96. $250 from May 18, 1891, to July 31, 1891, at 8%.

97. $500 from June 15, 1893, to Nov. 1, 1893, at 5%.

98. $475 from Dec. 25, 1894, to March 19, 1895, at 6%.

99. $1000 from May 4, 1895, to Oct. 10, 1895, at 7%.

100. $850 from Jan. 25, 1896, to June 30, 1896, at 8%.

101. In what time will any principal double itself at 8%?

102. In what time will any principal treble itself at 10%?

103. At what rate will any principal double itself in 12 yr.?

104. At what rate will any principal treble itself in 18 yr.?

143. Notes.

[FORM.]

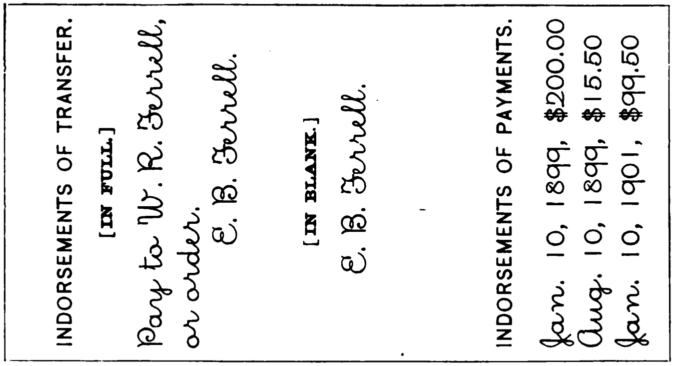
No. 12. Austin, Texas, July 10, 1898.

Three years after date, I promise to pay E. B. Ferrell, or order, at

THE FIRST NATIONAL BANK OF AUSTIN,

Five Hundred and no/100 Dollars, with interest at the rate of ten per cent. per annum until paid, for value received.

$500.00. S. A. Maroney.


INDORSEMENTS OF TRANSFER.

[IN FULL.]

Pay to W. R. Ferrell, or order.
E. B. Ferrell.

[IN BLANK.]

E. B. Ferrell.

INDORSEMENTS OF PAYMENTS.

Jan. 10, 1899, $200.00
Aug. 10, 1899, $15.50
Jan. 10, 1901, $99.50

A *Promissory Note* is a promise to pay to the party designated a certain sum of money, with or without interest, at the time specified in the note.

The *principal* is the amount of money mentioned in the note. The *Face* of a note is what it calls for on its face. If a note bears interest, the face includes principal and interest; if the note does not bear interest, the face is the principal.

NOTE.—Most authors define the *face of a note* as the principal only; but, if that be so, we should in the interest of simplicity discard the term *face*. (See Bank Discount.)

An Indorsement is something written on the back of a note. The above note is payable to *E. B. Ferrell*, or his order. He may order it paid to *W. R. Ferrell*; (see first indorsement above.) This is called an indorsement *in full*, because it is complete. A holder of a note may merely sign his name on the back; thus he orders it paid to any person who may be the holder at the time the note is due. This is called indorsement *in blank*.

NOTE.—There are other forms of indorsing a note for transfer, but they are not so commonly used as the above.

When payments are made upon a note, before final settlement, their dates and amounts are placed on the back of the note (see "Indorsements of payments" above).

In many States, the holder of a note or bill due at a future date must allow 3 days more than the time stated in the note for payment. These are called *Days of Grace*. The holder gets interest for this extra time, if the bill bears interest.

NOTE.—It is sometimes supposed that days of grace are only allowable in Bank Discount. *This is a mistake.* It is a law governing the payment of notes and bills, and not a principle of Bank Discount. But remember that it applies only to notes and bills, and not to all debts that may bear interest. The perplexing question to the student is, "*When am I to include days of grace and when not?*" If in this book a problem reads, "What is due Feb. 10, 1897?" or "Find the interest for 1 yr. 2 mo.," or "from May 1, 1900, to Jan. 1, 1901," do not add days of grace; but if it reads, "Dated Aug. 5, 1900, time 3 mo." (or 90 da.), or when the wording of the note is given, add days of grace. In business, this matter is governed by law or custom.

EXERCISE CXX.

1. Write a note that bears interest from date.

2. Why should a note be dated?

3. Who is the *maker* of your note?

4. Who is the *payee?*

5. Write a note that bears interest after it is due only.

6. What is the law concerning *days of grace* in your State?

7. If a note says nothing about interest, what rate of interest will it bear after it is due, in your State?

8. What is the highest rate of interest you can legally charge in your State?

9. If you charge more than is legal, what is it called?

10. What does the law say about *usury* in your State?

11. What is *indorsement in blank? in full?*

12. How are *payments* indorsed?

144. Partial Payments.—When partial payments are made on a note or debt bearing interest and running one year or less, it is customary to reckon interest on the principal and on each payment to the time of settlement, and subtract

the amount of the payments from the amount of the debt. This is called "*Merchants' Rule.*"

PRINCIPLES OF THE MERCHANTS' RULE: *1. The whole principal bears interest according to the terms of the note until final settlement.*

2. Each payment bears interest from the time it is made until date of final settlement.

When the debt or note bears *interest* for more than one year, the interest is computed by the "*U. S. Rule.*"

PRINCIPLES OF THE U. S. RULE: *1. Payments must first discharge interest due, and the balance, if any, will be applied to the discharge of the principal.*

2. Interest must not bear interest.

3. Payments must not bear interest.

EXAMPLES.

1. A note of \$400, dated Jan. 1, 1897, interest 6%, has the following payments: May 1, 1897, \$30; Nov. 16, 1897, \$100. What was due Jan. 1, 1898? (Merchants' rule. Why?)

Solution: (1) Interest on \$400 for 1 yr. at 6% = \$24.
(2) \$400 + \$24 = \$424, amt. of debt at maturity.
(3) From May 1, 1897, to Jan. 1, 1898 = 8 mo. = $\frac{2}{3}$ yr.
(4) Int. on \$30 for $\frac{2}{3}$ yr. at 6% = \$1.20.
(5) \$30 + \$1.20 = \$31.20, amt. of first payment.
(6) From Nov. 16, 1897, to Jan. 1, 1898 = 1 mo. 15 da. = $\frac{1}{8}$ yr.
(7) Int. on \$100 for $\frac{1}{8}$ yr. at 6% = \$.75.
(8) \$100 + \$.75 = \$100.75, amt. of second payment.
(9) \$424 − \$31.20 − \$100.75 = \$292.05, answer.

NOTE.—The work required in finding the interest in (1), (4) and (7) need not appear in the solution; but the teacher should insist upon *speed* and *accuracy* in mechanical work.

2. A note of \$150 is dated May 10, 1896; interest 6%. It

has the following indorsements: Sept. 10, 1897, $32; Sept. 10, 1898, $6.80. What is due Nov. 10, 1898? (U. S. Rule. Why?)

Dates.	*Payments.*
1898—11—10	
1898— 9—10	
1897— 9—10	
1896— 5—10	
1— 4— 0	.....$32
1— 0— 0	.....$6.80
2— 0	

NOTE.—Arrange the dates in order, with the latest date above. Begin below; subtract each date from the one next above it.

Solution: (1) Int. on $150 for $1\frac{1}{3}$ yr. at 6% = $12.
(2) $150 + $12 − $32 = $130, new principal.
(3) Int. on $130 for 1 yr. at 6% = $7.80.
(4) $7.80 − $6.80 = $1, unpaid interest.
(5) Int. on $130 for $\frac{1}{6}$ yr. at 6% = $1.30.
(6) $130 + $1 + $1.30 = $132.30, due at maturity.

EXERCISE CXXI.

1. A note for $820 was given June 12, 1897; interest 6%. It has the following indorsements: Aug. 12, 1897, $100; Nov. 12, 1897, $250; Jan. 12, 1898, $120. What was due Feb. 12, 1898?

2. A note of $1200, dated April 1, 1890, payable on demand, with interest at 7%, has the following indorsements: May 6, $210; July 5, $210; Oct. 18, $322. What was due Jan. 1, 1891?

3. A note of $2000, dated Jan. 4, 1893, interest 6%, has the following indorsements: Feb. 19, 1893, $400; June 29, 1894, $1000; Nov. 14, 1894, $520. What was due Dec. 24, 1896?

4. Find the amount due at maturity on note No. 12 on pages 290–1, including 3 days of grace (Kansas allows days of grace).

5. A note of $500, dated March 1, 1895, interest 6%, has the following indorsements: Sept. 1, 1895, $10; Jan. 1, 1896, $30; July 1, 1896, $11; Sept. 1, 1896, $80. What was due March 1, 1897?

145. Annual Interest. (See definition, p. 281.)

Example: A note of $800 drawing interest at 8%, payable

annually, runs for 5 years 3 months: find the amount due at maturity, if no payments have been made.

NOTE.—Each year's interest will bear interest after it becomes due, as follows:

1st year's int. for 4 yr. 3 mo.
2d year's int. for 3 yr. 3 mo.
3d year's int. for 2 yr. 3 mo.
4th year's int. for 1 yr. 3 mo.
5th year's int. for 0 yr. 3 mo.

Total...... 11 yr. 3 mo.

Therefore, in addition to the simple interest on the principal for 5 yr. 3 mo., we must count the interest on 1 year's interest for the sum of the above periods, or 11 yr. 3 mo.

Solution: (1) Int. on \$800 for 1 yr. at 8% = \$64, due annually.
(2) Int. on \$800 for 5¼ yr. at 8% = \$336, int. on principal.
(3) Int. on \$64 for 11¼ yr. at 8% = \$57.60, int. on int.
(4) \$800 + \$336 + \$57.60 = \$1193.60, amount due.

EXERCISE CXXII.

Find the annual interest—

1. On \$700 for 4 yr. 3 mo., at 6%, payable annually.
2. On \$840 for 2 yr. 4 mo., at 8%, payable semi-annually.
3. On \$1000 for 6 yr. 2 mo. 15 da., at 10%, payable annually.
4. On \$800 for 1 yr. 9 mo., at 5%, payable quarterly.
5. On \$400 for 6 yr. 4 mo. 8 da., at 10%, payable annually.

146. Compound Interest. (See definition, p. 281.)

EXAMPLES.

1. A note for \$800, drawing interest at 8%, payable annually, runs for 5 yr. 3 mo.: find the amount due at maturity by compound interest.

Solution: (1) Amt. of \$800 at 8% for 1 yr. = \$64 + \$800 = \$864.
(2) Amt. of \$864 at 8% for 1 yr. = \$69.12 + \$864 = \$933.12.
(3) Amt. of \$933.12 at 8% for 1 yr. = \$74.6496 = \$933.12 = \$1007.77.
(4) Amt. of \$1007.77 at 8% for 1 yr. = \$80.62 + \$1007.77 = \$1088.39.
(5) Amt. of \$1088.39 at 8% for 1 yr. = \$87.07 + \$1088.39 = \$1175.46.
(6) Amt. of \$1175.46 at 8% for ¼ yr. = \$23.51 + \$1175.46 = \$1198.97.

NOTE.—You will observe that this is the same problem which is solved by annual interest. Up to the end of the second year, the results obtained by annual interest and compound interest would be the same, but after that time the compound interest plan begins to advance **faster** than the annual interest plan. Why is this so?

Solving problems as above would be too tedious for use in business. A person having many problems to solve in compound interest should use a table like the following:

TABLE,

Showing the amount of $1, at 3, 4, 5, 6, 7 and 8 per cent., at Compound Interest, for any number of years, from 1 to 25.

Yr.	*3 per cent.*	*4 per cent.*	*5 per cent.*	*6 per cent.*	*7 per cent.*	*8 per cent.*
1	1.03	1.04	1.05	1.06	1.07	1.08
2	1.060	1.0816	1.1025	1.1236	1.1449	1.1664
3	1.092727	1.124864	1.157625	1.191016	1.225043	1.259712
4	1.125509	1.169859	1.215506	1.262477	1.310796	1.360488
5	1.159274	1.216653	1.276282	1.338226	1.402551	1.469328
6	1.194052	1.265319	1.340096	1.418519	1.500730	1.586874
7	1.229874	1.315932	1.407100	1.503630	1.605781	1.713824
8	1.266770	1.368569	1.477455	1.593848	1.718186	1.850930
9	1.304773	1.423312	1.551328	1.689479	1.838459	1.999004
10	1.343916	1.480244	1.628895	1.790848	1.967151	2.158924
11	1.384234	1.539454	1.710339	1.898299	2.104851	2.331638
12	1.425761	1.601032	1.795856	2.012196	2.252191	2.518170
13	1.468534	1.665074	1.885649	2.132928	2.409845	2.719623
14	1.512590	1.731676	1.979932	2.260904	2.578534	2.937193
15	1.557967	1.800944	2.078928	2.396558	2.759031	3.172169
16	1.604706	1.872981	2.182875	2.540352	2.952163	3.425942
17	1.652848	1.947900	2.292018	2.692773	3.158815	3.700018
18	1.702433	2.025817	2.406619	2.854339	3.379932	3.996019
19	1.753506	2.106849	2.526950	3.025600	3.616527	4.315701
20	1.806111	2.191123	2.653298	3.207135	3.869684	4.660957
21	1.860295	2.278768	2.785963	3.399564	4.140562	5.033833
22	1.916103	2.369919	2.925261	3.603537	4.430401	5.436540
23	1.973587	2.464716	3.071524	3.819750	4.740529	5.871463
24	2.032794	2.563304	3.225100	4.048935	5.072366	6.341180
25	2.093778	2.665836	3.386355	4.291871	5.427432	6.848475

2. Find the compound interest on $1500 for 12 yr. at 7%, payable annually.

Solution: (1) Comp. amt. of \$1 for 12 yr. at 7% = \$2.252191.
1500 × (1) = (2) Comp. amt. of \$1500 for 12 yr. at 7% = \$3378.2865.
(3) \$3378.2865 − \$1500 = \$1878.2865, answer.

3. Find the compound amount of \$400 for 4 yr. 2 mo. at 6% payable semi-annually.

NOTE.—Double the time and use ½ of the rate. Compound interest for 4 yr. 2 mo. at 6%, payable semi-annually, is the same as for 8 yr. 4 mo. at 3%, payable annually.

Solution: (1) Comp. amt. of \$1 at 3% for 8 yr. = \$1.26677.
(2) Comp. amt. of \$400 at 3% for 8 yr. = \$506.708.
(3) Int. on \$506.708 at 3% for 4 mo. = \$5.067.
(4) \$506.708 + \$5.068 = \$511.776, answer.

EXERCISE CXXIII.

Find the compound interest as in example 1—

1. On \$1000 for 4 yr. at 5%, payable annually.
2. On \$850 for 2 yr. at 4%, payable semi-annually.
3. On \$900 for 5 yr. 4 mo., at 6%, payable annually.

Find the compound interest as in examples 2 or 3—

4. On \$700 for 8 yr. at 5%, payable annually.
5. On \$1200 for 9 yr. at 8%, payable semi-annually.
6. On \$1100 for 6 yr. at 12%, payable quarterly.
7. Find the compound amount on \$1200 for 5 yr. 2 mo. at 6%, payable semi-annually.
8. Find the compound amount on \$600 for 17 yr. 3 mo. 15 da. at 10%, payable annually.

147. Bank Discount.—The *face* of a note is the amount due at maturity. If the note bears interest, the *face* is the amount of principal and interest due at maturity. If the note does not bear interest, the *face* is the principal. *Bank Discount* is the simple interest counted on the face of the note from the date of discounting to the date of maturity. The

manner of counting the interest is not different from that already studied. The *proceeds* are the difference between the bank discount and the face of the note. The "*time to run*" is the actual number of days from the date of discount to the date of maturity; this is sometimes called the "*term of discount*," which is perhaps a more appropriate name.

To find the date of maturity of a note: (1) When the time for which the note bears interest is expressed in months, as, "due in 4 mo.," count by months. From a certain day in one month to the same day in the next month is *1 mo.*, and so on. (2) When the time is expressed in days, count the actual days to find the date of maturity.

PRINCIPLES: *1. The face of the note, the amount due at maturity, is the amount to be discounted.*

2. The bank discount is the simple interest on the face of the note for the given time at the given rate.

3. The face of the note minus the bank discount equals the proceeds.

TERMS.

F, face of the note.
R, number of % of the discount.
T, time to run.
D, discount.
P, proceeds.

Developing the formulas for bank discount.

1. $D=\frac{FRT}{100}$. (Prin. 2.)

(1) $P=F-D$. (Prin. 3.)

(1) = (2) $P=F-\frac{FRT}{100}=\frac{100F-FRT}{100}$; or,

2. $P=\frac{F(100-RT)}{100}$.

NOTE.—In the following examples and exercise, *days of grace* are considered.

EXAMPLES.

1. A note of \$240, dated Jan. 1, 1888, time 4 months, was discounted March 5, 1888, at 10%: find the time to run, and proceeds.

Solution: (1) Date of maturity: 4 mo. 3 da. from Jan. 1, or May 4.

(2) Term of discount: From Mar. 5 to May 4, or 60 days. $= \frac{1}{6}$ yr.

(3) $P = \frac{240(100 - \frac{1}{6} \times 10)}{100} = 236.$

Proceeds, \$236.

2. A note of \$2400, dated July 3, 1891, bearing 5% interest, was discounted Jan. 9, 1892, at 10%; the note falls due 10 months from date. Find the time to run, bank discount, and proceeds.

Solution: (1) Date of maturity: 10 mo. 3 da. from July 3, May 6, 1892.

(2) Int. on \$2400 for $\frac{101}{120}$ yr. at 5% = \$101.

(3) \$2400 + \$101 = \$2501, face of note.

(4) Term of discount: From Jan. 9, 1892, to May 6, 1892, or 118 da. (leap year). $= \frac{59}{180}$ yr.

(5) $D = \frac{2501 \times 10 \times 59}{100 \times 180} = 81.98.$

Discount, \$81.98.

(6) $P =$ \$2501 − \$81.98 = \$2419.02, proceeds.

3. For what sum must a 60-day note, without interest, be drawn to produce \$500, when discounted for the full time at 6%?

Solution: (1) $P = \frac{F(100 - TR)}{100}.$

(2) $500 = \frac{F(100 - \frac{7}{40} \times 6)}{100}.$

(2) = (3) $500 = \frac{F \times 98\frac{19}{20}}{100}.$

(4) $F = \frac{500 \times 100 \times 20}{1979} = 505.306 -.$

Face, \$505.306.

Another Solution: (1) Bank discount on any principal for $\frac{7}{40}$ yr. at 6% = $1\frac{1}{20}$% of principal.

(2) Proceeds: 100% of prin. − $1\frac{1}{20}$% of prin. = $98\frac{19}{20}$% of prin.

(3) $98\frac{19}{20}$% of prin. = \$500.

(4) 1% of prin. = \$5.05306.

(5) 100% of prin. = \$505.306, answer.

EXERCISE CXXIV.

1. Commit to memory the principles of bank discount.

2. Develop the formulas, and give the relations.

Find date of maturity, time to run, and bank discount.

3. Prin., \$600; date, May 30; time, 5 mo.; rate, 10%; discounted, July 1.

4. Prin., \$840; date, Feb. 15; time, 8 mo.; rate, 6%; discounted, June 10.

5. Prin., \$1000; date, July 10; time, 90 da.; rate, 8%; discounted, Aug. 1.

Find date of maturity, time to run, and proceeds, if each of the following bears 8% interest from date:

6. Prin., \$800; date, Jan. 10; time, 6 mo.; rate, 10%; discounted, Mar. 6.

7. Prin., \$500; date, Apr. 12; time, 9 mo.; rate, 6%; discounted, Aug. 18.

8. Prin., \$470; date, May 15; time, 120 da.; rate, 10%; discounted, July 1.

9. I desire to use to-day \$680, which I can secure by giving a bankable note payable in 30 days, without interest, discounted at 9%. For what sum must I write the note?

148. True Discount.—The *Present Worth* of a debt payable at a future date without interest is that sum of money which, when put at interest for the given time, will amount to the debt. The difference between the present worth and the debt is the *True Discount.* To find the present worth of a debt is the same problem that we had in Interest: to find the principal when the amount of principal and interest, time, and rate are given. (See formula 2, p. 284.)

NOTE.—If the debt discounted be a note, given in a State where days of grace are recognized by law, *days of grace* should be considered in true discount as well as in any other application of interest.

If a debt or note, bearing interest, is discounted, the interest due at maturity must be added to the principal, before discounting.

EXAMPLES.

1. What is the present worth and true discount of a debt of \$627 if paid 9 months before it is due, discount 6%?

Solution: (1) $A = \frac{P(100+RT)}{100}$.

(2) $627 = \frac{P(100+6\times\frac{9}{12})}{100} = \frac{P\times 104\frac{1}{2}}{100}$.

(3) $P = \frac{627\times 100\times 2}{209} = 600$.

(4) *Present Worth* = \$600.

(5) *True Discount* = \$27.

No. 5. Chicago, Ill., Jan. 5, 1900.

Five months after date, I promise to pay —— Richard Roe, —— or order,

Twelve Hundred and no/100 —— Dollars,

with interest from date at the rate of 5 per cent. per annum, for value received.

\$ 1200.00. H. G. Adams.

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2. Find the true present worth of note No. 5, on April 9, 1900, discount, 6%.

Solution: (1) Interest on \$1200 at 5% for 5 mo. 3 da. = \$25.50.

(2) \$1200 + \$25.50 = \$1225.50.

(3) 5 mo. 3 da. from Jan. 5, 1900, is June 8, 1900, the date of maturity.

(4) From April 9 to June 8 = 60 da., the term of discount.

(5) $1225.50 = \frac{P(100+6\times\frac{1}{6})}{100}$.

(6) $P = \frac{1225.5\times 100}{101} = 1213.366$.

Answer, \$1213.366.

EXERCISE CXXV.

1. A debt of \$942 is due in 4 yr. 6 mo. 6 da., without interest: find its present worth if money is worth 6%.

2. A debt of \$273.75 is due in 1 yr. 7 mo., without interest: find its present worth and true discount at 6%.

3. A debt of \$366 is due in 3 yr. 8 mo., without interest: find its true discount at 6%.

4. A debt of \$338.20 is due in 5 yr. 7 mo. 27 da., without interest. What is its present worth, money being worth 10%?

5. A note of \$800, dated Jan. 10, time, 6 mo., bearing interest at 8%, is discounted at true discount Mar. 6, at 10%. Find the true discount.

6. The difference between true and bank discount is equal to the simple interest on the true discount for the given time at the given rate. Illustrate this fact by solving and comparing result in the following:

What is the difference between the true and bank discount on \$510 for 4 mo. at 6%?

149. Exchange.—Exchange is the process of making payment in distant places without transferring the money.

Exchange is effected by means of *drafts* or *bills of exchange.*

[A DRAFT.]

No. 9. FIRST NATIONAL BANK OF DES MOINES.

Des Moines, Iowa, July 6, 1895.

Pay to the order of Charles Clark, \$250.00

Two Hundred Fifty and ——— $\frac{00}{100}$ Dollars.

To WESTERN NATIONAL BANK, J. T. Bradley,

New York, N. Y. Cashier.

This is the form of an ordinary draft, or bill of exchange,

issued by one bank upon another. This draft purports to have been issued by the cashier of the First National Bank of Des Moines upon the Western National Bank of New York. The New York bank is hereby ordered to pay $250 to Charles Clark.

To illustrate the use of a draft: Suppose that you owe Charles Clark $250, and that he lives near New York city. One of the easiest as well as the safest ways to pay the debt is to go to your bank and get a draft on New York, just like the above, and send it to Clark. He can take it to the Western National Bank, or any other bank in that country, and get his money.

Banks usually charge a premium for issuing drafts, as, $\frac{1}{4}\%$, $\frac{1}{2}\%$, 1%, $1\frac{1}{2}\%$. Sometimes a bank will issue drafts at a discount; that is, give you something in order to get you to take the drafts. This is when there is a money panic at the money centers, or a scarcity of money at the local bank.

The above is a sight draft; it is payable on demand. Occasionally a customer wants a draft and does not care to have it paid for 30, 60, or 90 days. Such a draft is called a time draft, and the purchaser gets interest on his money for that time.

Domestic, or *Inland Exchange* is exchange between people in the same nation or country. *Foreign* exchange is exchange between people in different nations or countries.

PRINCIPLES: *1. The premium or discount is some number of per cent of the face value of the draft.*

2. In sight drafts, the cost is equal to the face of the draft plus the premium or minus the discount.

3. In time drafts, the cost is equal to the face of the draft minus the interest, plus the premium or minus the discount.

TERMS.

F, the face of the draft.

p, number of % of premium or discount.

P, the premium.

D, the discount.
C, the cost.
I, the interest.
R, number of % of the interest.
T, the time.

Development of formulas for exchange.

$$1.\quad P=\frac{pF}{100}.$$

$$2.\quad D=\frac{pF}{100}.$$

In sight drafts at a premium,

$$C=F+P=F+\frac{pF}{100}=\frac{100F+pF}{100};\ \text{or,}$$

$$3.\quad C=\frac{F(100+p)}{100}.$$

In sight drafts at a discount,

$$C=F-D=\frac{100F-pF}{100};\ \text{or,}$$

$$4.\quad C=\frac{F(100-p)}{100}.$$

In time drafts at a premium,

$$(1)\quad C=F-I+P=F-\frac{FRT}{100}+\frac{pF}{100}.$$

$$(1)=(2)\quad C=\frac{100F-FRT+pF}{100};\ \text{or,}$$

$$5.\quad C=\frac{F(100-RT+p)}{100}.$$

In time drafts at a discount,

$$C=F-I-P=F-\frac{FRT}{100}-\frac{pF}{100};\ \text{or,}$$

$$6.\quad C=\frac{F(100-RT-p)}{100}.$$

NOTE.— *Days of grace* **are usually considered in time drafts.**

EXAMPLES.

1. What will a draft on Kansas City for \$880 cost, at $\frac{1}{4}\%$ premium?

Solution: $C=\frac{880\times 100\frac{1}{4}}{100}=882.2.$

Answer, \$882.20.

2. What will a draft on St. Louis for \$1200 cost, at $\frac{1}{2}\%$ discount?

Solution: $C=\frac{1200\times 99\frac{1}{2}}{100}=1194.$

Answer, \$1194.

3. I have \$2000: how large a draft can I purchase with it, if I have to pay $\frac{1}{2}\%$ premium?

Solution: (1) $2000=\frac{F\times 100\frac{1}{2}}{100}.$

(2) $F=\frac{2000\times 100}{100\frac{1}{2}}=1990.05.$

Answer, \$1990.05.

4. How large a draft can I buy for \$2000, discount $\frac{1}{4}\%$?

Solution: (1) $2000=\frac{F\times 99\frac{3}{4}}{100}.$

(2) $F=\frac{2000\times 100}{99\frac{3}{4}}=2005.01+.$

Answer, \$2005.01+.

5. I want a 60-day draft on New York for \$2000: what will it cost me, if interest is 6% and premium on New York is $\frac{1}{2}\%$?

Solution: (1) $C=\frac{F\,(100-R\,T+p)}{100}.$

(2) $C=\frac{2000\,(100-\frac{7}{40}\times 6+\frac{1}{2})}{100}.$

(3) $C=\frac{2000\times 99\frac{9}{20}}{100}=\frac{2000\times 1989}{100\times 20}=1989.$

Answer, \$1989.

Another solution: (1) Int. on draft for 1 yr. at 6% = 6% of draft.

$\frac{7}{40}$ of (1) = (2) Int. on draft for $\frac{7}{40}$ yr. at 6% = $1\frac{1}{20}\%$ of draft.

(3) 100% of draft + $\frac{1}{2}\%$ of draft − $1\frac{1}{20}\%$ of draft = $99\frac{9}{20}\%$ of draft.

(4) 100% of draft = \$2000. (Basis.)

$\frac{1}{100}$ of (4) = (5) 1% of draft = \$20.

$99\frac{9}{20}\times$ (5) = (6) $99\frac{9}{20}\%$ of draft = \$1989, answer.

6. What will the draft in Example 5 cost if the exchange on New York is at a discount of $\frac{1}{2}\%$?

NOTE.—This is the same as above, except that you should use formula No. 6 instead of No. 5. Answer, $1969.

7. How large a 30-day draft can I get for $2000, interest being 8% and premium $\frac{1}{4}\%$?

$$\textit{Solution:}\ (1)\ 2000=\frac{F(100-8\times\frac{11}{120}+\frac{1}{4})}{100}.$$

$$(1)=(2)\ 2000=\frac{F\times 99\frac{31}{60}}{100}=\frac{F\times 5971}{100\times 60}$$

$$(3)\ F=\frac{2000\times 100\times 60}{5971}=2009.71+.$$

Answer, $2009.71+.

8. What will be the size of the draft in Example 7, if exchange is at a discount of $\frac{1}{4}\%$?

NOTE.—This is the same as above, except that you should use formula No. 6 instead of No. 5. Answer, $1980.52+.

9. What will an exchange on London for £240 cost, at $4.85?

Solution: (1) Cost of £1 = $4.85. (Basis.)
240 × (1) = (2) Cost of £240 = $1164.

10. I have $2635.20 with which to buy a draft on Liverpool, at $4.88: what will be the size of the draft?

Solution: (1) $4.88 = cost of £1. (Basis.)

$\frac{1}{4.88}$ of (1) = (2) $1 = cost of £$\frac{1}{4.88}$.

2635.2 × (2) = (3) $2635.20 = cost of £540.

EXERCISE CXXVI.

1. Define exchange, domestic exchange, foreign exchange.

2. Draw a draft, using amount, names, date, etc., as you may choose.

3. Develop all the formulas for exchange.

4. Commit to memory the principles.

5. Give the relations expressed by the formulas.

6. What will be the cost of a draft on Chicago for \$8600 at $\frac{1}{2}\%$ premium?

7. What will be the cost of a draft on New Orleans for \$9400 at $\frac{1}{2}\%$ discount?

8. I paid \$5420 for a draft on Kansas City at $\frac{3}{8}\%$ premium: what was the size of the draft?

9. I paid \$7240 for a draft on New York at $\frac{1}{4}\%$ discount: what was the face of the draft?

10. What will be the cost of a 60-day draft on San Francisco for \$6540, premium $\frac{1}{4}\%$, interest 8%.

11. What will be the cost of a 90-day draft on Philadelphia for \$2450, discount $\frac{1}{2}\%$, and interest 6%?

12. I have \$7800 with which I desire to purchase a 60-day draft on New York: if exchange is at $1\frac{1}{4}\%$ premium, and money is worth 8%, what will be the face of my draft?

13. What size bill of exchange can I get on London for \$12150, if exchange is \$4.86?

14. I bought a bill of exchange on Liverpool for £842: what did it cost if exchange is \$4.83½?

15. A draft on Pittsburg cost \$1771, exchange $\frac{3}{8}\%$ premium: what was the face of the draft?

16. The face of a 30-day draft on St. Paul is \$4220: what did it cost if exchange is $1\frac{1}{2}\%$ premium and interest 8%?

17. A grain merchant in Toledo sold 11875 bu. of corn at 40¢ per bushel; deducting 3% commission, he purchased a 60-day draft with the proceeds, interest 6%, premium 2%: required the face of the draft.

18. An agent owed his principal \$1011.84. He bought a draft with the sum, and remitted: the principal received \$992. Find the rate of exchange.

19. What is the cost of a bill on Amsterdam for 4800 guilders, quoted at 41½¢, brokerage ½?

20. Jones, of St. Louis, has a debt of \$6000 to pay in New York. The direct exchange is $\frac{1}{2}\%$ premium; but exchange on Philadelphia is $\frac{1}{4}\%$ premium, and from Philadelphia to New York is $\frac{1}{4}\%$ discount. By circular exchange how much will pay the debt, and how much does he gain over the direct?

C. MECHANICS.

150. Force.—*Force* is that which tends to produce or overcome motion of matter.

The several units of force in common use are the *poundal*, *dyne*, *pound*, and *gram*, and may be defined as follows:

1. The force required to add 1 ft. in 1 sec. to the velocity of 1 lb. of matter = 1 poundal (pl.).

2. The force required to add 1 cm. in 1 sec. to the velocity of 1 g. of matter = 1 dyne.

3. The force required to add 32.16 ft. in 1 sec. to the velocity of 1 lb. of matter = 1 pound (N. Y.).

4. The force required to add 980 cm. in 1 sec. to the velocity of 1 g. of matter = 1 gram (N. Y.).

NOTE.—It should be remembered that we have pounds of matter and pounds of force, and grams of matter and grams of force. The pupil should distinguish clearly between the two.

The units which have "N. Y." after them are called *gravity units*, because they vary with the attraction of gravitation at different places on the earth. As given above, the units are correct at the sea-level at New York. The *dyne* and *poundal* are called *absolute* units, and are the same for all places.

TABLE OF EQUIVALENTS.

1 lb. of force (N. Y.) = 32.16 pl.
1 g. of force (N. Y.) = 980 dynes.
1 pl. = 13826 dynes.

EXERCISE CXXVII.

1. Reduce 534 pl. to dynes.
2. Reduce 340 lb. (N. Y.) to pl.
3. Reduce 34 g. (N. Y.) to dynes.

4. Reduce 103695 dynes to pl.
5. Reduce 19296 pl. to lb. (N. Y.).
6. Reduce 3859200 dynes to lb. (N. Y.).
7. Reduce 1960 lb. (N. Y.) to g. (N. Y.).
8. Reduce 34 lb. (N. Y.) to dynes.

The amount of velocity added per second to the motion of a body is called its *Acceleration*.

TERMS.

F, the force.
M, the mass.
v, the velocity,
t, the time (in sec.).
a, the acceleration.

Development of the formulas for force.

If a body starts to move from a state of rest, its velocity at the end of the 1st sec. will be the same as its acceleration; or

(1) Velocity for 1 sec. $= a$.
and (2) Velocity for t sec. $= at$; or,

$$1.\ v = at.$$

From the definition of a poundal,

(1) Force required to add 1 ft. in 1 sec. to the velocity of 1 lb. of matter $= 1$ pl.

(2) Force required to add v ft. in 1 sec. to the velocity of 1 lb. of matter $= v$ pl.

(3) Force required to add v ft. in 1 sec. to the velocity of M lb. of matter $= Mv$ pl.

(4) Force required to add v ft. in t sec. to the velocity of M lb. of matter $= \frac{Mv}{t}$ pl.; or, stated abstractly,

$$2.\ F = \frac{Mv}{t}.$$

From formula 1,

$$(1)\ a = \frac{v}{t}.$$

Substituting a for $\frac{v}{t}$ in formula 2, we have—

$$3.\ F = Ma.$$

When M is expressed in lb., v or a in ft., and t in seconds, formulas 2 and 3 give the force in *poundals*. When M is expressed in g., v or a in cm., and t in sec., these formulas give the force in *dynes*.

Since a *poundal* is $\frac{1}{32.16}$ pound of force (N. Y.), and a *dyne* is $\frac{1}{980}$ gram of force (N.Y.), these formulas will represent $\frac{1}{32.16}$ as many *pounds* as *poundals;* or $\frac{1}{980}$ as many *grams* as *dynes*.

Now, 32.16 ft. = 980 cm.

If we represent this amount by g, these formulas, when used to find the force in *pounds* or *grams*, will be—

$$4.\ F=\frac{Mv}{gt},\ \text{and}$$

$$5.\ F=\frac{Ma}{g}.$$

NOTE.—Do not forget that g in *feet* is *32.16;* in *centimeters* it is *980.*

EXAMPLES.

1. What force will give 15 lb. of matter an acceleration of 25 ft. per second?

Solution: $F=15\times 25=375$.
∴ the required force is 375 poundals.

2. What force will give to 21 grams of matter an acceleration of 10 centimeters per second?

Solution: $F=21\times 10=210$.
∴ the required force is 210 dynes.

3. What force will give 6 cwt. 20 lb. of matter an acceleration of 1 ft. 6 inches per second?

Solution: (1) 6 cwt. 20 lb. = 620 lb.
(2) 1 ft. 6 in. = $1\frac{1}{2}$ ft.
(3) $F=620\times 1\frac{1}{2}=930$.
∴ the required force = 930 poundals.

4. What is the force in poundals that will in 10 minutes produce a velocity of a mile a minute in 100 lb.?

Solution: (1) 1 mi. = 5280 ft.
(2) 1 min. = 60 sec.
(3) 10 min. = 600 sec.
(4) Velocity of 1 mi. per min. = velocity of 88 ft. per sec.
(5) $F = \frac{100 \times 88}{600} = 14\frac{2}{3}$.
∴ the reqd. force is $14\frac{2}{3}$ poundals.

5. A force of 30 dynes, acting for 12 seconds upon a body, gives it a velocity of 120 cm. per second. Find the mass of the body.

Solution: (1) $30 = \frac{120 \times M}{12} = 10 \times M$.
(2) $M = \frac{30}{10} = 3$.
∴ the required mass is 3 grams.

6. A force of 40 poundals is acting upon a mass of 16 pounds; how much velocity is given the body per second?

NOTE.—Acceleration is wanted.

Solution: (1) $40 = 16 \times a$.
(2) $a = \frac{40}{16} = 2\frac{1}{2}$.
∴ the reqd. acceleration is $2\frac{1}{2}$ ft. per sec.

7. A body, acted upon by a force of 100 dynes, receives an acceleration of 20 cm. per second. Find its mass.

Solution: (1) $100 = 20 \times M$.
(2) $M = \frac{100}{20} = 5$.
∴ the reqd. mass is 5 grams.

8. A force of 360 poundals acts upon a mass of 6 pounds for 4 seconds. What velocity did the body receive?

Solution: (1) $360 = \frac{6 \times v}{4} = \frac{3 \times v}{2}$.
(2) $v = \frac{360 \times 2}{3} = 240$.
∴ the reqd. velocity is 240 ft. per sec.

9. A body of 30 pounds received a velocity of 480 ft. per second from a force of 120 poundals. How long did the force act to do this?

Solution: (1) $120 = \frac{480 \times 30}{t}$.

(2) $120 \times t = 480 \times 30$.

(3) $t = \frac{480 \times 30}{120} = 120$.

$\therefore$ the reqd. time is 120 sec., or 2 min.

10. How many pounds of force will add to 321.6 pounds of matter a velocity of 20 ft. per second?

Solution: $F = \frac{321.6 \times 20}{32.16} = 200$.

$\therefore$ the reqd. force is 200 lb.

11. A force of 800 pounds gives a body a velocity of 80 ft. per second in 5 seconds. Find its weight.

Solution: (1) $800 = \frac{80 \times M}{32.16 \times 5}$.

(2) $M = \frac{800 \times 32.16 \times 5}{80} = 1608$.

$\therefore$ the reqd. weight is 1608 lb.

12. Find the acceleration per second given to a body of 700 grams by a force of 140 grams.

Solution: (1) $140 = \frac{700 \times a}{980}$.

(2) $a = \frac{140 \times 980}{700} = 196$.

$\therefore$ the reqd. acceleration is 196 cm. per sec.

EXERCISE CXXVIII.

1. What is force? Velocity? Acceleration?
2. What is a poundal?
3. What is a dyne?
4. What is a lb. of force (N. Y.)?
5. What is a gram of force (N. Y.)?

6. Repeat the table of equivalents.

7. How would you reduce grams to pl.?

8. How would you reduce pounds to dynes?

9. What is the difference between a pound of force and a pound of matter?

10. Develop the formulas for force.

11. Give the relation expressed by each formula.

12. A constant force acting upon a mass of 15 g. for 4 sec. gives it a velocity of 20 cm. per sec. Find the force in dynes.

13. What is the force in poundals that will give to 1000 lb. of matter in 20 min. a velocity of 2 miles a minute?

NOTE.—Express the time in sec. and the velocity in ft. What is the velocity per sec., if the mass is traveling at the rate of a mile a minute?

14. A mass of 15 lb. lying on a smooth, flat table is acted upon by a force of 60 pl. Find the velocity at the end of 2 sec.

15. A force of 1000 dynes acting on a mass for 1 sec. gives a velocity of 20 cm. per sec. Find the mass in grams.

16. What force in dynes will in 1 sec. give to a mass of 12 g. a velocity of 6 cm. per sec.?

17. What velocity will be produced by a force of 490 dynes acting on a mass of 70 g. for 10 sec.?

18. In what time will a force of 7500 pl. give to a mass of 300 lb. a velocity of 250 ft. per sec.?

151. Work.—Work is moving a body; and the amount of work depends upon the amount of force (resistance) overcome and the distance through which that force is overcome.

The several units of work in common use are the *foot-poundal* (ft.-pl.), *erg*, *foot-pound* (ft.-lb.), and *gram-centimeter* (g.-cm.), and are defined as follows:

1. The work done in overcoming 1 poundal of force through a distance of 1 foot = 1 foot-poundal.

2. The work done in overcoming 1 dyne of force through a distance of 1 centimeter = 1 erg.

3. The work done in overcoming 1 pound of force through a distance of 1 foot = 1 foot-pound.

4. The work done in overcoming 1 gram of force through a distance of 1 centimeter = 1 gram-centimeter.

NOTE.—The *foot-poundal* and the *erg* are called *absolute units* of work; and the *foot-pound* and *gram-centimeter* are called *gravity units*. This classification agrees with that of the force units (see p. 308).

TABLE OF EQUIVALENTS.

1 g.-cm. (N. Y.) = 980 ergs.
1 ft.-lb. (N. Y.) = 32.16 ft.-pl.
1 ft.-pl. = 421402 ergs.

EXERCISE CXXIX.

1. Reduce 16 ft.-pl. to ergs.
2. Reduce 132 ft.-lb. to ft.-pl.
3. Reduce 842 g.-cm. to ergs.
4. Reduce 8428040 g.-cm. to ft.-pl.
5. Reduce 64320 ft.-pl. to ft.-lb.
6. Reduce 69580 ergs to g.-cm.
7. Reduce 742 ft.-lb. to ergs.
8. Reduce 21070100 ergs to ft.-pl.

TERMS.

W, the work.
F, the force (resistance).
d, distance.

Developing the formula for work.

From the definition of foot-poundal,

(1) Work done in overcoming 1 pl. of force through a distance of 1 ft. $=1$ ft.-pl.

(2) Work done in overcoming F pl. of force through a distance of 1 ft. $=F$ ft.-pl.

(3) Work done in overcoming F pl. of force through a distance of d ft. $=Fd$ ft.-pl.; or, stated abstractly,

$$W=Fd.$$

NOTE.—When F is poundals and d, ft., the work is *foot-poundals;* when F is dynes and d, cm., the work is *ergs;* when F is pounds and d, ft., the work is *foot-pounds;* and when F is grams and d, cm., the work is *gram-centimeters.*

EXERCISE CXXX.

1. Define work, foot-poundal, foot-pound, erg, and gram-centimeter.

2. Does the question of time enter into the work formula? Is the work in No. 4 the same whether done in 10 min. or in 10 hr.?

3. Develop the work formula, and give the relation which it expresses.

4. A team of horses draws a wagon 2 miles. If the draft is 1200 pl., how much work is performed?

5. 240000 ergs of work is performed in overcoming a force of 800 dynes. Through what distance did the body move?

6. 9640 ft.-pl. of work overcomes a certain force through a distance of 120 ft. 6 in. Find the force.

7. Two horses draw a machine 40 rd., and thus do 105600 ft.-lb. of work. What is the draft of the machine in pounds? in poundals?

8. How much work is done in lifting 50 Kg. of matter 20 meters high? (Answer in g.-cm.)

9. 6240 g.-cm of work is performed in overcoming a resistance of 120 dynes. Find the distance.

152. Activity or Power.—The rate at which work is

done is called *Power* or *Activity*, and its amount depends upon the *work done* and the *time* in which it is done.

There are two units of activity in use, the *watt* and the *horse-power*, which may be defined as follows:

1. The activity required to do 10000000 ergs of work in 1 second = 1 watt.

2. The activity required to do 550 ft.-lb. of work in 1 second = 1 horse-power (h.-p.).

NOTE.—The *watt* is called the *absolute unit* of activity, and the *horse-power* the *gravity unit.*

EQUIVALENTS.

1 h.-p. (N. Y.) = 746 watts.

TERMS.

A, the activity or power.
W, the work.
t, the time.

Development of formulas for activity.

From the definition of watt,

(1) Activity reqd. to do 10000000 ergs of work in 1 sec. = 1 watt.

(2) Activity reqd. to do 1 erg of work in 1 sec. = $\frac{1}{10000000}$ watt.

(3) Activity reqd. to do W ergs of work in 1 sec. = $\frac{W}{10000000}$ watts.

(4) Activity reqd. to do W ergs of work in t sec. = $\frac{W}{10000000t}$ watts; or, stated abstractly,

$$1.\quad A=\frac{W}{10^7\times t}.$$

This formula is correct for finding the activity in watts. By using the definition for *horse-power*, the formula for finding horse-powers will be found to be—

$$2.\quad A=\frac{W}{550\times t}.$$

EXERCISE CXXXI.

1. What is *power* or *activity*? How does it differ from *work*?

2. Define *watt*, and *horse-power*.

3. Reduce 520 h.-p. to watts.

4. Reduce 17158 watts to h.-p.

5. Develop all the formulas for activity. Give the relation expressed by each.

6. How many watts of activity will do 720000000 ergs of work in 6 sec.?

7. How much work will be done by an activity of 12 h.-p. in $\frac{1}{2}$ min.?

8. How much time will be required for an activity of 5 watts to do 150000000 ergs of work?

9. How many h-p. of activity will do 385000 ft.-lb. of work in 5 sec.?

10. What is the horse-power of an engine that can raise 2376 lb. 1000 ft. in 3 min.?

11. How far can a 2-horse-power engine raise 5 tons in 30 sec.?

12. How long will it take a 2-horse-power engine to raise 5 tons 100 ft.?

153. Simple Machines.—The *simple* machines are the *Lever*, *Wheel and Axle*, *Inclined Plane*, and *Screw*. (See Dictionary or Physics for definitions.)

I. The elements involved in *lever* problems are *power* (P), weight (W), the distance from the fulcrum to the power (D), and the distance from the fulcrum to the weight (d).

NOTE.—The distances D and d, are often spoken of as *power arm* and *weight arm*, respectively.

LAW OF THE LEVER: *The power is to the weight as the weight arm is to the power arm.*

Proportion: $P : W :: d : D$.

II. The elements involved in the problems of the *wheel and axle* are power (P), weight (W), radius (diameter or circumference) of the wheel (R), and the radius (diameter or circumference) of the axle (r).

Law of the Wheel and Axle: *The power is to the weight as the radius (diameter or circumference) of the axle is to the radius (diameter or circumference) of the wheel.*

Note.—In this statement of the law, the power is supposed to be applied to the wheel.

Proportion: $P : W :: r : R.$

Note.—The diameters or the circumferences may be used instead of the radii.

III. The elements involved in problems of the *inclined plane* are *power* (P), *weight* (W), the *height* of the plane (h), the *length* of the plane (l), and the *base* of the plane (b).

Laws of the Inclined Plane: (1) When the power acts parallel to the plane, *The power is to the weight as the height of the plane is to its length.* (2) When the power acts horizontally, *The power is to the weight as the height of the plane is to its base.*

Proportions: (1) $P : W :: h : l.$
(2) $P : W :: h : b.$

IV. The elements involved in problems of the *screw* are power (P), weight (W), the distance between the threads of screw (d), and the circumference through which the power moves (C).

Law of the Screw: *The power is to the weight as the distance between the threads is to the circumference through which the power moves.*

Proportion: $P : W :: d : C.$

Note.—These proportions are formulas, stated in the form of proportion instead of that of equations.

EXAMPLES.

1. The pilot-wheel of a boat is 3 feet in diameter; the axle, 6 inches; the resistance of the rudder 180 pounds. What power applied to the wheel will move the rudder?

$$6 \text{ in.} = \tfrac{1}{2} \text{ ft.}$$

Proportion: (1) $P : 180 :: \tfrac{1}{2} : 3.$

(2) $P = \dfrac{180 \times \frac{1}{2}}{3} = 30.$

∴ the power will be 30 lb. to balance the resistance of the rudder; anything more than 30 lb. will *move* it.

2. The distance between the threads of a screw is $1\tfrac{1}{2}$ inches, the circumference through which the power moves is $13\tfrac{1}{2}$ feet. What weight will a power of 20 pounds support (neglecting friction)?

$$13\tfrac{1}{2} \text{ ft.} = 162 \text{ in.}$$

Proportion: (1) $20 : W :: 1\tfrac{1}{2} : 162.$

(2) $W = \dfrac{20 \times 162}{1\frac{1}{2}} = 2160.$

∴ the required weight is 2160 lb.

3. A man exerts 120 pounds of power in supporting a 280-lb. barrel upon a 7-ft. plane extending from the ground to a wagon: how high is the wagon (pressure exerted in the direction of plane)?

Proportion: (1) $120 : 280 : h : 7.$

(2) $h = \dfrac{120 \times 7}{280} = 3.$

∴ the height is 3 ft.

4. The length of a lever is 16 feet; weight at one end, 100 pounds: what power must be put at the other, 4 feet from the fulcrum, to support the weight?

Proportion: (1) $P : 100 :: 12 : 4.$

(2) $P = \dfrac{100 \times 12}{4} = 300.$

∴ the required power is 300 lb.

EXERCISE CXXXII.

Name the elements and give the law of—

1. The *lever.*

2. The *wheel and axle.*

3. The *inclined plane.*

4. The *screw.*

5. State the formula (proportion) for each.

6. Suppose a power of 75 pounds be applied to one end of a 12-foot lever to support a load at the other end: what will be the load when the fulcrum is at the center? when the fulcrum is 3 feet from the weight?

7. In a nut-cracker the nut is placed 1 inch from the hinge (fulcrum), the hand 6 inches. If I exert a pressure of 10 pounds, how many pounds of resistance does the nut furnish?

8. Four men hoist an anchor weighing 1 ton; the barrel of the capstan is 8 inches in diameter; the circle described by the handspikes is 6 ft. 8 in. in diameter. How great a force must each man exert?

9. A power of 70 pounds on a wheel 10 feet in diameter balances a weight of 300 pounds: find the diameter of the axle.

10. The base of an inclined plane is 10 feet, the height 3 feet. What force applied parallel to the base will balance a weight of 2 tons?

11. In No. 10, if the force applied be 500 pounds, what weight will it support?

12. A weight of 800 pounds rests on an inclined plane 8 feet high, being held in equilibrium by a force of 25 pounds, acting parallel with the plane: find the length of the plane.

13. How great a pressure will be exerted by a power of 15 pounds applied to a screw whose head is 1 inch in circumference, and whose threads are $\frac{1}{8}$ inch apart?

D. MISCELLANEOUS.

154. Partitive Proportion.—Partitive Proportion is the process of dividing a given number into parts having a given ratio to one another.

EXAMPLES.

1. Divide $25 into two parts, in the ratio of 2 to 3.

NOTE.—If the two answers were obtained and the first answer divided into two parts and the second answer into 3 parts, there would then be 5 parts all of the same size. These 5 parts then represent the amount to be divided, or $25; 2 parts, the first answer; and 3 parts the second answer. Hence,

Proportions: (1) 5 parts : 2 parts :: $25 : $( )?
(2) 5 parts : 3 parts :: $25 : $( )?
∴ the answers are $10 and $15.

2. A lever 33 feet long has a power of 50 pounds at one end and a weight of 60 pounds at the other end. Where is the fulcrum, if the power balances the weight?

From the law of the lever, we know that—

50 lb.: 60 lb.:: wt. arm : pr. arm.

Then the weight arm must be to the power arm as 50 to 60, or 5 to 6. Now, the problem is to divide 33 feet into 2 parts in the ratio of 5 to 6.

Proportions: (1) 11 parts : 5 parts :: 33 ft.: ( )?
(2) 11 parts : 6 parts :: 33 ft.: ( )?
∴ the arms are 15 ft. and 18 ft.

3. Divide 65 bushels of wheat into 3 parts, in the ratio of 3 to 2 to $1\frac{1}{2}$.

$3 \text{ pt.} + 2 \text{ pt.} + 1\frac{1}{2} \text{ pt.} = 6\frac{1}{2} \text{ pt.}$

Proportions: (1) $6\frac{1}{2}$ parts : 3 parts :: 65 bu.: ( ) bu.?
(2) $6\frac{1}{2}$ parts : 2 parts :: 65 bu.: ( ) bu.?
(3) $6\frac{1}{2}$ parts : $1\frac{1}{2}$ parts :: 65 bu.: ( ) bu.?
∴ the reqd. parts are 30 bu., 20 bu., and 15 bu.

EXERCISE CXXXIII.

1. Divide \$750 into 2 parts in the ratio of 1 to 4; in the ratio of 4 to 11.

2. Divide \$220 in the ratio of 3 to 7 to 12.

3. A and B balance on opposite ends of a pole 10 feet long. A weighs 120 pounds, and B 180 pounds: where is the fulcrum?

4. At the opposite ends of a lever 20 feet long, two forces are acting whose sum is 1200 pounds. The two arms of the lever are as 2 to 3. What are the two forces when the lever is in equilibrium?

5. A fails, and can pay his three creditors, E, F, and G, but \$750. He owes E \$700, F \$500, and G \$300: how much should each receive?

155. Mixture Problems.—

1. Different kinds of tea sold as follows: 10 lb. for \$7, 30 lb. for \$18, and 20 lb. for \$8. Find the average price per pound.

Solution: (1) Price of 10 lb. = \$7.
(2) Price of 30 lb. = \$18.
(3) Price of 20 lb. = \$8.
(1)+(2)+(3) = (4) Price of 60 lb. = \$33.
$\frac{1}{60}$ of (4) = (5) Price of 1 lb. = \$.55.

2. A mixture composed of sugar at 2¢ per lb. and sugar at 5¢ per lb. is worth 3¢ per lb. How is it mixed?

NOTE.—There is a *gain* on the 2¢ sugar when it becomes a part of a mixture valued at 3¢ per lb., and a *loss* on the 5¢ sugar. They must be mixed in such a ratio as will make the *gain* on the one equal to the *loss* on the other.

Solution: (1) Gain on 1 lb. of 2¢ sugar at 3¢ = 1¢.
(2) Loss on 1 lb. of 5¢ sugar at 3¢ = 2¢.
(2) = (3) Amt. of 5¢ sugar reqd. to lose 2¢ = 1 lb.
$\frac{1}{2}$ of (3) = (4) Amt. of 5¢ sugar reqd. to lose 1¢ = $\frac{1}{2}$ lb.
(1) = (5) Amt. of 2¢ sugar reqd. to gain 1¢ = 1 lb.

Comparing (5) and (4), we see that the mixture must be made in the ratio of 1 lb. of the 2¢ sugar to $\frac{1}{2}$ lb. of the 5¢ sugar, or in the ratio of 2 to 1.

3. How many pounds of coffee at 25¢ per lb. must I mix with 15 lb. at 30¢ per lb. to make a mixture worth 28¢?

Solution: (1) Loss on 1 lb. of 30¢ coffee at 28¢ = 2¢.
$15 \times (1) = (2)$ Loss on 15 lb. of 30¢ coffee at 28¢ = 30¢, whole loss.
(3) Gain on 1 lb. of 25¢ coffee at 28¢ = 3¢.
$(3) = (4)$ Amt. of 25¢ coffee reqd. to gain 3¢ = 1 lb.
$10 \times (4) = (5)$ Amt. of 25¢ coffee reqd. to gain 30¢ = 10 lb., answer.

4. Mix two kinds of coffee at 20¢ and 25¢ per lb., to make a mixture of 25 lb. worth 24¢ per lb.

Solution: (1) Gain on 1 lb. of 20¢ coffee at 24¢ = 4¢.
(2) Loss on 1 lb. of 25¢ coffee at 24¢ = 1¢.
$(1) = (3)$ Amt. of 20¢ coffee reqd. to gain 4¢ = 1 lb.
$\frac{1}{4}$ of $(3) = (4)$ Amt. of 20¢ coffee reqd. to gain 1¢ = $\frac{1}{4}$ lb.
$(2) = (5)$ Amt. of 25¢ coffee reqd. to lose 1¢ = 1 lb.

From (4) and (5), we see the mixture must be in the ratio of $\frac{1}{4}$ lb. of 20¢ coffee to 1 lb. of the 25¢ coffee, or the ratio of 1 to 4. The problem now is, to divide 25 lb. in the ratio of 1 to 4.

(6) 5 parts : 1 part :: 25 lb. : ( )?
(7) 5 parts : 4 parts :: 25 lb. : ( )?

$$\therefore \text{ Smaller part} = \frac{1 \times 25 \text{ lb.}}{5} = 5 \text{ lb.}$$

$$\text{and the larger part} = \frac{4 \times 25 \text{ lb.}}{5} = 20 \text{ lb.}$$

EXERCISE CXXXIV.

1. How many pounds of sugar at 8¢ per lb. must I mix with 10 lb. at 11¢ per lb. to make a mixture worth 10¢ per lb.?

2. How must I mix coffee at 24¢ per lb. with coffee at 30¢ per lb. to make a mixture worth 26¢ per lb.?

3. I mix sugar at 9¢ per lb. with 72 lb. at 5¢; the whole mixture is worth 7¢ per lb.: how much of the 9¢ sugar did I use?

4. A vintner mixed two kinds of wine, one at 40¢ per pint and the other at 60¢ per pint. He finds that the mixture is worth just 55¢ per pint: how did he mix it?

5. A saloon-keeper watered his 60¢ wine, and sold it at 45¢ per pint: if there were 6 gallons and 1 quart of water added, how much wine was there?

6. A man mixed coffee at 26¢ per lb. with coffee at 32¢ per lb., making a mixture of 48 lb. worth 30¢ per lb.: how much of each did he use?

7. I mix wine at 40¢ and 50¢ per pint with wine at 60¢ per pint; the 50¢ and 60¢ wine is mixed in the ratio of 2 of the 50¢ to 1 of the 60¢: if the mixture is worth 48¢ per pint, how do I mix it?

8. If quarters and dimes are mixed in such a ratio that the same money might be molded into the same number of 15¢ pieces, how are they mixed?

156. Equation of Payments.—Equation of Payments is the process of finding the date on which one payment may be made in settlement of several debts or an account of several items, falling due at different times, without loss to either debtor or creditor.

EXAMPLES.

1. I owe $600 due in 10 months, $800 due in 9 months, and $1000 due in 6 months, without interest. When can I settle all together?

Solution: Suppose that I should settle all *now*, then I would lose—

(1) Use of $600 for 10 mo. = use of $1 for 6000 mo.
(2) Use of $800 for 9 mo. = use of $1 for 7200 mo.
(3) Use of $1000 for 6 mo. = use of $1 for 6000 mo.

$2400, amt. paid. Use of $1 for 19200 mo., loss.

(4) Use of $1 for 19200 mo. = use of $2400 for 8 mo.
∴ 8 months from *now*, $2400 will settle the debt.

2. My account with Jones is as follows: I owe him $600 due in 11 months, $800 due in 4 months, $500 due in 1 year, and $200 due now; he owes me $700 due in 8 months and $600 due

in 9 months. When will \$800 settle the account? and how much will be due 1 year hence, interest being 6%?

Solution: Assume *now* as the date of settlement.

(1) Use of \$600 for 11 mo. = use of \$1 for 6600 mo.
(2) Use of \$800 for 4 mo. = use of \$1 for 3200 mo.
(3) Use of \$500 for 12 mo. = use of \$1 for 6000 mo.
(4) \$200 due now.

\$2100, amt. paid. Use of \$1 for 15800 mo., loss.

(5) Use of \$700 for 8 mo. = use of \$1 for 5600 mo.
(6) Use of \$600 for 9 mo. = use of \$1 for 5400 mo.

\$1300, amt. recd. Use of \$1 for 11000 mo., gain.

(7) \$2100 − \$1300 = \$800, balance.
(8) Use of \$1 for 15800 mo. − use of \$1 for 11000 mo. = use of \$1 for 4800 mo., net loss.
(9) Use of \$1 for 4800 mo. = use of \$800 for 6 mo.
∴ the time of payment is 6 mo. from *now*. (1st ans.)
(10) Amt. of \$800 at 6% for ½ yr. = \$24 + \$800 = \$824. (2d ans.)

3. I bought goods Jan. 1, 1898, on 30 days' credit, \$600; Feb. 10, on 60 days' credit, \$500. When can I settle the whole by paying \$1100?

Solution: Assume Jan. 1st as the date of settlement. The first debt will be paid 30 days before it is due, and the second will be paid 100 days before it is due.

(1) Use of \$600 for 30 days = use of \$1 for 18000 days.
(2) Use of \$500 for 100 days = use of \$1 for 50000 days.

\$1100, amt. paid. Use of \$1 for 68000 days, loss.

(3) Use of \$1 for 68000 days = use of \$1100 for 62 days, nearly.
62 days after Jan. 1, 1898, is March 4, 1898.

NOTE.—A part of a day is always counted as a full day in equating the time of payment. This is plain, for in the above problem, as the time was more than 61 days, it must fall on the 62d day.

These solutions are given in full. After the pupil writes out

a few in this way, he should shorten the solution. Thus, for Example 1:

Solution: (1) $600 \times 10 = 6000.$
(2) $800 \times 9 = 7200.$
(3) $\underline{1000} \times 6 = \underline{6000}.$
$2400 \qquad 19200.$
$19200 \div 2400 = 8.$
Answer, \$2400, 8 mo. from *now*.

EXERCISE CXXXV.

1. I owe \$400 due in 4 months and \$800 due in 7 months, each bearing interest at 8%. Find the equated time and the amount due.

2. I owe Smith \$1200 due in 8 months and \$600 due in 5 months; Smith owes me \$2000 due in 10 months: when will \$200 settle what he owes me?

3. A merchant owes his creditor for goods, as follows: July 3, \$640; July 15, \$300; Aug. 10, \$520: when will \$1460 settle his account?

4. In No. 3, a note of \$500, due Aug. 1st without interest, is credited. When will \$960 settle the balance?

5. If, in No. 3, each amount is a note due in 60 days, bearing 10% interest, find the equated time of payment and the amount due.

157. Partnership.

EXAMPLES.

1. A, B and C form a partnership. A puts in \$200, B \$300, and C \$400. They gain \$450. Divide it among them.

Solution: The whole capital is \$900. Each partner's capital is to the whole capital as his share of the gain is to the whole gain. Hence—

(1) For A, \$200 : \$900 : : \$( ) : \$450 ?
∴ A's gain is \$100.

(2) For B, \$300 : \$900 : : \$( ) : \$450 ?
∴ B's gain is \$150.

(3) For C, \$400 : \$900 : : \$( ) : \$450 ?
∴ C's gain is \$200.

2. A invests \$100 for 6 months; B, \$200 for 2 months. They gain \$50. Divide it between them.

Solution:

(1) Earnings of \$100 for 6 mo. = earnings of \$600 for 1 mo., A's part.

(2) Earnings of \$200 for 2 mo. = earnings of \$400 for 1 mo., B's part.

(3) The whole earnings = earnings of \$1000 for 1 mo.

(4) For A, $\left\{\begin{array}{c}\text{Earnings of}\\ \$600\text{ for 1 mo.}\end{array}\right\} : \left\{\begin{array}{c}\text{earnings of}\\ \$1000\text{ for 1 mo.}\end{array}\right\} :: \$(\ ) : \$50?$

∴ A's gain is \$30.

NOTE.—The "1 mo." may be omitted, as it is the same in both the 1st and the 2d terms, and the proportion may be shortened as in (5).

(5) For B, \$400 : \$1000 : : \$( ) : \$50?

∴ B's gain is \$20.

EXERCISE CXXXVI.

1. Three persons purchase a farm for \$2800, of which A pays \$1200, B \$1000, and C \$600. How shall they divide the rent, which pays \$224 a year?

2. A, B and C owned a cargo of corn valued at \$3475.60. A owned $\frac{1}{4}$, B $\frac{1}{8}$, and C the remainder. How much did each lose if the cargo, which was insured for \$2512, was lost?

3. A, B and C form a copartnership. B puts in \$250 for 6 months, C \$275 for 8 months, and A \$450 for 4 months. Divide their gain, \$825.

4. A, B and C form a copartnership for 1 year, and invest respectively \$9600, \$8400, and \$7200. After 4 months, A invests \$2000 additional, B \$1400, and C \$800. They gain \$12800: what is each man's gain?

158. Longitude and Time.—I. LONGITUDE. **Longitude** is the distance east or west of the prime meridian measured in degrees (°), minutes (′), and seconds (″).

TABLE.

$$1° = 60'.$$
$$1' = 60''.$$

The meridian of Greenwich, England, has been selected as the prime meridian. 180° east from the prime meridian (0°) reaches half around the earth. Any place on that half of the earth is said to be in *east longitude.* 180° from 0° west reaches around the other half of the earth. Any place on that half of the earth is said to be in *west longitude. The longitude of a place is its distance (less than 180°) east or west of the prime meridian.* The distance measured in one direction (east or west) around the earth is 360°.

II. TIME.—(1) *Sun Time:* The sun appears to travel westward around the earth (360°) once in 24 hours. It passes over 15° of longitude in 1 hour of time; 15′ of longitude in 1 minute of time; and 15″ of longitude in 1 second of time. Therefore the *difference of longitude* between two places, expressed in ° ′ ″, is numerically *15 times* the *difference of time*, expressed in *hours, minutes, seconds.* Of two places on meridians nearer to each other than 180°, the *place west* is *earlier* and the *one east* is *later*, except when the I. D. L. (see below) and midnight are both between the two places; in which case the *place west* is *later* by the difference between their *difference of time* and *24 hours.*

(*2*) *The Day:* Every day begins first at or near the 180th meridian. On this account the 180th meridian is often called the *International Date Line* (I. D. L.).* When midnight is on the I. D. L., there is one, and but one, day on all the earth, (say) Sunday. As midnight moves westward, it changes the day on every part of the earth over which it passes to the new day, Monday. When it has gone all the way around and again

*The International Date Line used to have a very winding course. After coming southward through Bering Strait, it took a southwesterly course, passing west of the Philippine Islands, then bending eastward, then southward into the Southern ocean near the 180th meridian. But now, after passing southward through Bering Strait, it swerves a little (20° or 30°) to the westward. Returning to the 180th meridian about 40° N., it follows that meridian to a point about 10° S. Then it swerves eastward about 10° or 15°. Returning to the 180th meridian again about 50° S., it continues southward on that meridian. This description was taken from a diagram given in *The Pathfinder* in 1899.

reaches the I. D. L., it is Monday all over the earth. When any place is between the I. D. L. and the midnight meridian, if it is west of the I. D. L. and east of midnight, it is the *new day;* if it is east of the I. D. L. and west of midnight, it is the *old day.*

(*3*) *Standard or Railroad Time:* There are five standards of railroad time in the United States and Canada: Intercolonial, Eastern, Central, Mountain, and Pacific. Each covers a longitude of 15°. Intercolonial commences 52½° W., and the Pacific ends 127½° W. The time throughout each standard is the sun time of its central meridian: For Intercolonial, 60th; for Eastern, 75th; for Central, 90th; for Mountain, 105th; and for Pacific, 120th.

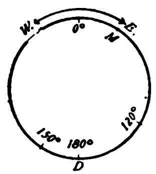
NOTE.—Railroad companies vary the boundary-lines of these standards in some places.

III. PROBLEMS: There are two classes of problems usually given in Longitude and Time: (1) Given the longitudes of two places and the time at one of them, to find the time at the other; and (2) given the times of two places and the longitude of one, to find the longitude of the other.

EXAMPLES.

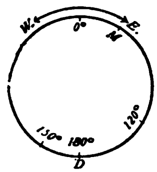
1. When it is noon, Monday, 150° W., what is the time 120° E.?

DIRECTION.—In drawing a figure to represent a longitude problem, the following will help to make your problem clear: (1) Sketch a circular figure to represent a parallel of latitude; (2) place 0° at the top; (3) 180° (I. D. L.) at the bottom; (4) locate approximately the two places under consideration and midnight (M.).



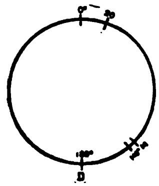
D. = I. D. L.
M. = midnight.

Solution : (1) $180° - 150° = 30°$.
(2) $180° - 120° = 60°$.
(3) $30° + 60° = 90°$, difference of longitude.
$\therefore$ 6 hours = difference of time.
6 hours earlier than noon = 6 a. m.
150° W. is in the old day (Monday).
120° E. is in the new day (Tuesday).



2. When it is 6 hr. 40 min. p. m., Friday, 20° E., what is the time 140° E.?

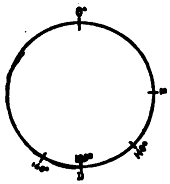
Solution : (1) $140° - 20° = 120°$, difference of longitude.
$\therefore$ 8 hr. = difference of time.
8 hr. later than 6 hr. 40 min. p. m. = 2 hr. 40 min. a. m. of the following day, Saturday.



3. When it is 7 hr. 20 min. a. m., Friday, 160° W., on what meridian is it 3 hr. 20 min. a. m. Saturday?

Solution: (1) 7 hr. 20 min. − 3 hr. 20 min. = 4 hr., difference of time.
$\therefore$ 60° = difference of longitude.

Since 3 hr. 20 min. a. m. is earlier than 7 hr. 20 min. a. m., the required place is *west*. 60° west of 160° W. is 140° E.; and it is Saturday, because it is in the new day (beyond the I. D. L.).



4. When it is noon at Greenwich, 0°, what is the time at St. Louis (standard time)?

Solution : The time of St. Louis is that of 90° W., or 6 hr. earlier than noon = 6 hr. a. m. of the same day.

5. When it is 3 o'clock a. m. at Kansas City, where is it 2 o'clock a. m.? 4 o'clock a. m.?

Solution : (1) Kansas City is in Central Time; it is 2 o'clock a. m. on all the standard west, or $7\frac{1}{2}°$ each side of 105° W.
(2) It is 4 o'clock a. m. on all the standard east of Central Time, or $7\frac{1}{2}°$ each side of 75° W.

EXERCISE CXXXVII.

1. When it is 9 o'clock p. m., Wednesday, 20° E., what is the time at Greenwich (0°)?

2. When it is 9 o'clock p. m., Wednesday, 20° E., what is the time at 30° W.?

3. When it is 9 o'clock p. m., Wednesday, 20° E., what is the time at 120° W.?

4. When it is 9 o'clock p. m., Wednesday, 20° E., what is the time at 80° E.?

5. When it is noon at Greenwich, at what place is it 2 o'clock a. m.?

6. When it is noon at Greenwich, at what place is it 7 o'clock a. m.?

7. When it is noon at Greenwich, at what place is it 3 o'clock p. m.?

8. When it is noon at Greenwich, at what place is it 8 o'clock p. m.?

9. When it is 1 hr. 20 min. p. m., Saturday, 110° E., what is the time 170° E.?

10. When it is 1 hr. 20 min. p. m., Saturday, 110° E., what is the time 105° W.?

11. When it is 1 hr. 20 min. p. m., Saturday, 110° E., at what place is it 9 hr. p. m.?

12. When it is 1 hr. 20 min. p. m., Saturday, 110° E., at what place is it 11 hr. p. m.?

13. When it is 1 hr. 20 min. p. m., Saturday, 110° E., what is the time at Boston? at New York? at St. Louis? at Cincinnati? at Kansas City? at San Francisco (standard time)?

14. When it is noon on the 180th meridian, it is 6 hr. 52 min. 40 sec. p. m. at Harrisburg, Pa. (sun time). Find the longitude of Harrisburg.

15. A watch is set right at Pekin, China, longitude 116° 26′ E. On what meridian is the watch when, at noon, it shows 9 hr. 20 min. a.m.?

16. When it is 5 minutes after 4 o'clock on Sunday morning at Honolulu, longitude 157° 52′ W., what is the time at Sydney, Australia, longitude 151° 11′ E.?

17. The battle of Manila began 41 min. after 5 o'clock, Sunday morning, May 1. Manila is in longitude 121° 20′ E. Washington is in longitude 77° W. What was the time at Washington when the battle began, by sun time? by standard time?

18. When it is 10 o'clock p. m. Wednesday, 20° E., what is the time at Sydney, Australia? at Honolulu? railroad time at San Francisco, Cal.? at St. Louis, Mo.? at Boston, Mass.?

NOTE.—When time is required, always give the day as well as the hour of the day.

PART III.

I. STUDY OF NUMBERS.

A. INTEGRAL NUMBERS.

159. Positive and Negative Numbers.—The signs plus (+) and minus (−) are placed before numbers, when written with figures or letters, to indicate opposite meanings or conditions. Thus,

If +$4 means $4 *gain*, −$4 means $4 *loss*.
If +$4 means $4 *capital*, −$4 means $4 *debt*.
If +4 means 4 *to be added*, −4 means 4 *to be subtracted*.
If +4° means 4°(temperature) *above zero*, −4° means 4° *below zero*.
If +4 ft. means 4 ft. traveled to the *right*, −4 ft. means 4 ft. traveled to the *left*.

A **Positive Number** is a number which, when written, is preceded by the *plus sign*, expressed or implied.

NOTE.—For convenience, a positive number, standing alone or first in an expression, is usually written without the sign, +.

A **Negative Number** is a number which, when written, is preceded by the *minus sign*.

NOTE.—The sign, −, is never omitted before negative numbers.

All numbers are either *positive* or *negative*. To illustrate:

5, 4, 3, 2, 1, 0, −1*, −2, −3, −4, −5.

The numbers in this series decrease in absolute value from left to right; each number, beginning with 4, is one less than

* Read, *minus one*, *minus two*, and so on.

the number to the left of it. Notice that from 0 to the right the absolute value decreases as the numerical value increases; -5 is less than -4, -4 is less than -3, etc.

The question is sometimes asked, "Can a number be less than nothing?" Yes: a number may be less than nothing. Suppose that I have but \$5 and owe nothing, and you ask me how much I am worth: I would say, "\$5." That means that I am worth \$5 *more than nothing*, +\$5. Suppose now, that I have the \$5, and in addition I owe some one \$3. I am now worth \$3 *less than before*, or but \$2. Suppose again that I owe another debt of \$3: my worth is again decreased \$3, and I am now worth \$1 *less than nothing*, −\$1. If I should have another debt of \$3, I would then be worth \$4 *less than nothing*, −\$4.

NOTE.—It does not follow that all negative numbers represent value less than nothing. A *negative distance* is not less than nothing; $-10°$ (temperature) is not less than nothing.

160. Numbers Expressed by Means of Letters.—In expressing numbers by means of letters, or letters and figures, there are four things to be considered: (1) the literal part, (2) the coefficient, (3) the exponent, and (4) the sign.

The *Literal Part* is the letter or letters used as factors in expressing a number. Thus,

In $-5a^2$, a is the literal part; in $5a^2b$, a and b form the literal part.

The *Coefficient* is a factor (expressed by figures in this book), placed on the left of the literal part. Thus,

In $15ax$, *15* is the coefficient.

NOTE.—In such expressions as ax, or $-ax^2$, the coefficient is not expressed, but is understood to be 1.

An *Exponent* is a small figure (sometimes a letter), placed

just above and to the right of a factor to show what power of the factor is used. (See p. 179.)

The *Sign* of a number is used to indicate whether the number is positive or negative.

A number of one term is called a *Monomial.* A number of more than one term is called a *Polynomial.*

A polynomial of two terms is called a *Binomial;* of three terms, a *Trinomial.*

EXERCISE CXXXVIII.

1. Give use of the signs plus and minus.

2. What is a *positive number?*

3. What is a *negative number?*

4. Explain how some numbers do have values *less than nothing.*

5. How many things are to be considered in expressing numbers by letters or by letters and figures?

6. Explain what is meant by *literal part.* Give example.

7. Define *coefficient.* Give example.

8. Define *exponent.* Give example.

9. What does the *sign of a number* show?

10. Define *monomial.* Give example.

11. Define *polynomial.* Give example.

12. Define *binomial.* Give example.

13. Define *trinomial.* Give example.

161. Addition.—Since we now have to consider both positive and negative numbers, our process of addition must be extended to cover the addition of negative as well as positive numbers.

In explaining the following examples, let us assume that $+$ means *capital* and $-$ means *debt.*

EXAMPLES.

1. Add $+\$8$ and $+\$4$.

Process: $+\$8+(+\$4)=+\$12$, result.

$4 (capital) added to $8 (capital) makes $12 (capital).

2. Add $-\$8$ and $-\$4$.

Process: $-\$8+(-\$4)=-\$12$, result.

$4 (debt) added to $8 (debt) makes $12 (debt).

3. Add $+\$8$ and $-\$4$.

Process: $+\$8+(-\$4)=+\$4$, result.

$4 (debt) added to $8 (capital) gives $4 (capital).

4. Add $-\$8$ and $+\$4$.

Process: $-\$8+(+\$4)=-\$4$, result.

$4 (capital) added to $8 (debt) gives $4 (debt).

Let us write these processes abstractly and compare:

(1) $+8+(+4)=+12$.
(2) $-8+(-4)=-12$.
(3) $+8+(-4)=+4$.
(4) $-8+(+4)=-4$.

Observe that in (1) the signs were alike (both plus), and we added 8 and 4; in (2) the signs were alike (both minus), and we added 8 and 4; in (3) and (4) the signs were unlike (one plus, the other minus), and we subtracted 4 from 8. In every case, the sign of the result is the same as that of the larger addend.

RULE FOR ADDITION: *When the signs are alike, add the numbers and prefix the common sign; when unlike, subtract the smaller from the larger, giving the result the sign of the larger addend.*

5. Add 50, $+38$, -14, -73, -52, $+23$, $+44$.

I. *Process:* (1) $50+38=88$; (2) $88-14=74$; (3) $74-73=1$; (4) $1-52=-51$; (5) $-51+23=-28$; (6) $-28+44=16$, result.

NOTE.—The process may be much shortened by uniting the positive numbers into one amount and the negative numbers into another. The difference of these amounts preceded by the sign of the greater will be the mathematical sum. Thus:

II. *Process:* (1) $50+38+23+44=155$.
(2) $-14-73-52=-139$.
(3) $155-139=16$, result.

Literal numbers are similar, when they are composed of the same letters affected by the same exponents. Thus:

ab^2 and $25\,ab^2$ are similar, but
a^2b and ab^2 are dissimilar.

PRINCIPLES: *1. Only similar numbers can be added to form one term.*

2. The addition of dissimilar numbers is indicated by placing between them the proper sign.

6.	7.	8.
$-4\,a$	$5\,xy$	$7\,x^2y$
$-7\,a$	$4\,xy$	$-5\,x^2y$
$+5\,a$	$-9\,xy$	$-14\,x^2y$
$+15\,a$	$-8\,xy$	$+11\,x^2y$
$-\,a$	$-3\,xy$	$+\,x^2y$
$+8\,a$, result.	$-11\,xy$, result.	0, result.

9. Add $-7b$, $+9b$, $+6xy$, $+4b$, $-7xy+z$.

Process:
$$\begin{array}{l} -7b+6xy+z \\ +9b-7xy \\ \underline{+4b\qquad\qquad} \\ 6b-xy+z, \text{ result.} \end{array}$$

10. Add $12xy^2-15ax-c$, $7ax+9c$, $-5xy^2+17ax-12c$.

Process:
$$\begin{array}{rrr} 12xy^2 & -15ax & -c \\ & +7ax & +9c \\ \underline{-5xy^2} & \underline{+17ax} & \underline{-12c} \\ 7xy^2 & +9ax & -4c, \text{ result.} \end{array}$$

11. Add $5(x-y)$, $-7(x-y)$, $15(x-y)$.

Process:
$$\begin{array}{r} 5(x-y) \\ -7(x-y) \\ \underline{15(x-y)} \\ 13(x-y), \text{ result.} \end{array}$$

NOTE.—When the expressions in parentheses are similar they may be treated as one number.

As, $5\times No.-7\times No.+15\times No.=13\times No.$

EXERCISE CXXXIX.

1. Give *rule for addition.* Give principles.

2. Explain examples 1 to 4, letting the numbers represent distance,— the positive numbers distance measured to the right, and negative numbers distance measured to the left.

Thus, In Example 1,

$$\begin{array}{l} +8, \text{ to the right.} \\ \underline{+4,} \text{ to the right.} \\ +12; \text{ or, 12 to the right of starting-point.} \end{array}$$

In Example 3,

$$\begin{array}{l} +8, \text{ to the right.} \\ \underline{-4,} \text{ to the left.} \\ +4; \text{ or, still 4 to the right of starting-point.} \end{array}$$

Add (orally):

3.	*4.*	*5.*	*6.*	*7.*
$\begin{array}{r} -7 \\ -5 \\ \hline \end{array}$	$\begin{array}{r} +7 \\ +5 \\ \hline \end{array}$	$\begin{array}{r} +7 \\ -5 \\ \hline \end{array}$	$\begin{array}{r} -7 \\ +5 \\ \hline \end{array}$	$\begin{array}{r} 14 \\ -9 \\ \hline \end{array}$

8.	*9.*	*10.*	*11.*	*12.*
$\begin{array}{r} -5 \\ +7 \\ -4 \\ \hline \end{array}$	$\begin{array}{r} +9 \\ -6 \\ +3 \\ \hline \end{array}$	$\begin{array}{r} -17 \\ +4 \\ -2 \\ \hline \end{array}$	$\begin{array}{r} +4 \\ -18 \\ +5 \\ \hline \end{array}$	$\begin{array}{r} -13 \\ +12 \\ -6 \\ \hline \end{array}$

13.	*14.*	*15.*	*16.*	*17.*
$\begin{array}{r} -a \\ +5a \\ -7a \\ +a \\ \hline \end{array}$	$\begin{array}{r} -5x \\ +25x \\ -3x \\ -11x \\ \hline \end{array}$	$\begin{array}{r} +12xy \\ -15xy \\ +9xy \\ -12xy \\ \hline \end{array}$	$\begin{array}{r} -3a^2x \\ +5a^2x \\ -12a^2x \\ +4a^2x \\ \hline \end{array}$	$\begin{array}{r} -20x^2y^2 \\ +18x^2y^2 \\ +4x^2y^2 \\ -14x^2y^2 \\ \hline \end{array}$

18.	*19.*	*20.*
$\begin{array}{r} 7(x-2y) \\ -9(x-2y) \\ +(x-2y) \\ \hline \end{array}$	$\begin{array}{r} -8(x^2-y) \\ +6(x^2-y) \\ -12(x^2-y) \\ \hline \end{array}$	$\begin{array}{r} 9(x+y)^2 \\ -5(x+y)^2 \\ +(x+y)^2 \\ \hline \end{array}$

Copy and add:

21. $12a-9c,\ -4a+17c,\ -7c+15a,\ 27a-c.$

22. $42a^2-5ab,\ -36a^2+12ab,\ 7a^2-15ab,\ -ab-14a^2.$

23. $a^2-2ab+b^2,\ 5a^2-9ab+4b^2,\ -12a^2+26ab-12b^2,\ 36a^2-24b^2+7ab.$

24. $11a^2b+11ab^2+a^2b^2,\ 5a^2b-21ab^2+6a^2b^2,\ 14ab^2-5a^2b^2+9a^2b,\ -4a^2b^2+5ab^2+3a^2b.$

25. $9x^2y+4x+3y-9y^2,\ 4y^2-2x^2y-5x+3y,\ 12x-14y-6y^2-2x^2y,\ x^2y-x+y-y^2.$

162. Subtraction.—It will help to explain the process of subtracting positive and negative numbers, if we will note the fact, *that any monomial may be expressed as a binomial, one of whose terms is positive and the other negative, and that the numerical value of either term may be increased at will, provided the numerical value of the other be correspondingly increased.*

EXAMPLES.

1. Represent $+5$ by an expression containing $+8$, $+12$, $+17$, -5, -14, -21.

(1) $5=8-3.$	(4) $5=10-5.$
(2) $5=12-7.$	(5) $5=19-14.$
(3) $5=17-12.$	(6) $5=26-21.$

2. Represent -7 by an expression containing $+4$, $+12$, $+15$, -9, -13, -25.

(1) $-7=-11+4.$	(4) $-7=-9+2.$
(2) $-7=-19+12.$	(5) $-7=-13+6.$
(3) $-7=-22+15.$	(6) $-7=-25+18.$

The process of subtraction consists in taking from the minuend the number expressed by the subtrahend.

EXAMPLES.

3. From $+8$ take $+4$.

(1) $+8-(+4)=+4$, result. Taking $+4$ from $+8$ leaves $+4$.

4. From -8 take $+4$.

(1) $-8=-12+4$.	The -8 is the same as $-12+4$; taking away the $+4$ leaves -12.
(2) $(-12+4)-(+4)=-12$, result.	

5. From $+8$ take -4.

(1) $+8=+12-4$.	The $+8$ is the same as $+12-4$; taking away the -4 leaves $+12$.
(2) $(+12-4)-(-4)=+12$, result.	

6. From -8 take -4.

(1) $-8-(-4)=-4$, result. Taking -4 from -8 leaves -4.

NOTE.—The same result will be obtained by the *capital* and *debt* plan of explanation. (See Addition, p. 336.)

Let us compare these examples:

(1) $+8-(+4)=+\ 4$.
(2) $-8-(+4)=-12$.
(3) $+8-(-4)=+12$.
(4) $-8-(-4)=-\ 4$.

Observe that, when the signs of the subtrahend and minuend were alike (in (1) and (4)), we took the difference between the 8 and the 4; when the signs of the subtrahend and minuend were unlike (in (2) and (3)), we took the sum of the 8 and the 4.

As we might have expected, this is exactly the reverse of our comparison in addition.

Suppose that in the four examples above, we change the sign of each subtrahend and add (instead of subtracting) and see what results we get:

(1) $+8+(-4)=+4$.
(2) $-8+(-4)=-12$.
(3) $+8+(+4)=+12$.
(4) $-8+(+4)=-4$.

These results are just the same as above, and we have subtracted by changing the signs of the subtrahends and adding.

RULE FOR SUBTRACTION: *Conceive the sign of the subtrahend to be changed, (the $+$ to $-$ or the $-$ to $+$) and proceed as in addition.*

NOTE.—Do not actually change the sign of the subtrahend on paper

or slate in written work, but do it *mentally*. An actual change of written signs leads to confusion in reviewing the work.

PRINCIPLES: *1. The result can be expressed in one term only when the minuend and subtrahend are similar.*

2. The subtraction of dissimilar numbers is indicated by placing between them the proper sign.

7. From $12-5+4$ take $8-3+2$.

(1) $12-5+4=11$.
(2) $8-3+2=7$.
(3) $11-7=4$.

Or, $12-5+4-8+3-2=4$.

Collect the terms of both minuend and subtrahend; change the sign of the subtrahend, and proceed as in addition. Or, change all signs of the subtrahend, and then collect.

8. $5+17-3-(4+8-10+2)=?$

Process: $5+17-3-(4+8-10+2)=$
$5+17-3-4-8+10-2=15$, result.

NOTE.—The $-$ before the parentheses shows that all the numbers within are to be subtracted, and that the expression within the parentheses is a subtrahend.

RULE FOR CLEARING OF PARENTHESES PRECEDED BY A MINUS SIGN: *Drop the parentheses with the minus sign, and change the sign of every term within.*

9.	*10.*	*11.*	*12*
$7a$	$-xy$	$+3x^2$	$-17xy^2$
$5a$	$+5xy$	$-9x^2$	$-5xy^2$
$2a$, result.	$-6xy$, result.	$+12x^2$, result.	$-12xy^2$, result.

13. $7x^2+4ax+2b^2-(-5b^2+7xa-5x^2)=(\ \ )$?

Process:
$$\begin{array}{r} 7x^2+4ax+2b^2 \\ -\ \ 5x^2+7ax-5b^2 \\ \hline 12x^2-3ax+7b^2 \end{array}$$, result.

Or,

$$7x^2+4ax+2b^2-(-5b^2+7ax-5x^2)=$$
$$7x^2+4ax+2b^2+5b^2-7ax+5x^2=$$
$$12x^2-3ax+7b^2$$, result.

14. From $5(a+x)$ take $2(a+x)$.

$$\begin{array}{l} 5(a+x) \\ \underline{2(a+x)} \\ 3(a+x), \text{ result.} \end{array}$$

EXERCISE CXL.

1. Explain examples 3 to 6, letting positive numbers mean *capital* and negative numbers mean *debt;* letting positive numbers mean distance moved to the *right*, and negative numbers distance moved to the *left*.

2. Give the *rule for subtraction.*

3. Give the *principles.*

4. Give the *rule for removing parentheses preceded by a minus sign.*

5. Do you change any signs in removing parentheses preceded by a plus sign ?

Subtract (orally):

6.	*7.*	*8.*	*9.*	*10.*
-7	$+7$	$+7$	-7	12
$\underline{-5}$	$\underline{+5}$	$\underline{-5}$	$\underline{+5}$	$\underline{-9}$

11.	*12.*	*13.*	*14.*	*15.*
$-a$	$-7a$	$-9a$	$-12x$	$+12x$
$\underline{-5a}$	$\underline{+3a}$	$\underline{-4a}$	$\underline{-6x}$	$\underline{+6x}$

16.	*17.*	*18.*	*19.*	*20.*
$-a^2x$	$-9x^2y$	$-12mn$	$+ab$	$+17x^3$
$\underline{7a^2x}$	$\underline{+12x^2y}$	$\underline{-9mn}$	$\underline{-20ab}$	$\underline{-3x^3}$

21.	*22.*	*23.*
$-9(x-2y)$	$+5(x^2-y^2)$	$7(x^2+2xy+y^2)$
$\underline{+4(x-2y)}$	$\underline{+9(x^2-y^2)}$	$\underline{-4(x^2+2xy+y^2)}$

Copy and subtract:

24. From $3a-5b^2-c$ take $7a+4b^2-10c$.

25. From $7a^2+9xy-6b^3$ take $11b^3+4xy-10a^2$.

26. From $-75(x^2y-xy^2)$ take $-150(x^2y-xy^2)$.

27. $17x-5y+6-(5x+11y-10)=(\ \ )$?

28. $9xy-(7ax+12xy-3z)=(\ \ )$?

29. $7x^2y-3xy^2+y^3-(-7y^3+6xy^2-9yx^2)=(\ \ )$?

30. $a+b-c-d-x-(a-b+c-d-x)=(\ \ )$?

31. $x^2+2xy+y^2-(x^2-2xy+y^2)=(\ \ )$?

32. $x^3+3x^2y+3xy^2+y^3-(x^3-3x^2y+3xy^2-y^3)=(\ \ )$?

163. Multiplication.—When the multiplier is positive, it shows how many times the multiplicand is to be taken as an addend (see p. 21); but, when negative, it shows how many times the multiplicand is to be taken as a *subtrahend*.

EXAMPLES.

1. Multiply $+7$ by $+3$.

NOTE.—Starting with 0, we *add* $+7$ three times.

Process: (1) $0+(+7)=+7$, used once.
(2) $+7+(+7)=+14$, used twice.
(3) $+14+(+7)=+21$, used three times.
$\therefore\ +3\times+7=+21$, result.

2. Multiply -7 by $+3$.

NOTE.—Starting with 0, *add* -7 three times.

Process: (1) $0+(-7)=-7$, used once.
(2) $-7+(-7)=-14$, used twice.
(3) $-14+(-7)=-21$, used three times.
$\therefore\ +3\times-7=-21$, result.

3. Multiply $+7$ by -3.

NOTE.—Starting with 0, *subtract* $+7$ three times.

Process: (1) $0-(+7)=-7$, used once.
(2) $-7-(+7)=-14$, used twice.
(3) $-14-(+7)=-21$, used three times.
$\therefore\ -3\times+7=-21$, result.

4. Multiply -7 by -3.

NOTE.—Starting with 0, subtract -7 three times.

Process: (1) $0-(-7)=+7$, used once.
(2) $+7-(-7)=+14$, used twice.
(3) $+14-(-7)=+21$, used three times.
$\therefore\ -3\times-7=+21$.

Let us compare these examples:

(1) $+3\times+7=+21$.
(2) $+3\times-7=-21$.
(3) $-3\times+7=-21$.
(4) $-3\times-7=+21$.

Observe (1) that the numerical products are the same (each 21, or the product of 3 and 7), and (2) when the signs of the factors are alike (both plus as in (1) or both minus as in (4)), the product is *positive;* but when the signs are unlike (as in (2) and (3)), the product is *negative.*

LAW OF SIGNS IN MULTIPLICATION: *Like signs give plus; unlike signs give minus.*

5.	*6.*	*7.*	*8.*
$+12a$	$+12a$	$-12a$	$-12a$
$+6b$	$-6b$	$+6b$	$-6b$
$+72ab$, result.	$-72ab$, result.	$-72ab$, result.	$+72ab$, result.

9. $-7a\times+5b\times-3c=(\ \ )$?

Process: (1) $-7a\times+5b=-35ab$.
(2) $-35ab\times-3c=+105abc$, result.

10. Multiply $4m-7n$ by $5n$.

Process:
$$\begin{array}{r} 4m-7n \\ -5n \\ \hline -20mn+35n^2;\ \text{or,} \\ 35n^2-20mn,\ \text{result.} \end{array}$$

NOTE.—In writing polynomials, it is customary (but not necessary) to write a positive term (if there be one) on the left, and omit its sign.

11. Multiply $5a(x+y)$ by $-b$.

NOTE.—This multiplication may be performed (1) by multiplying the $5a$ by $-b$, (2) by multiplying $(x+y)$ by $-b$, or (3) by removing the parentheses from the multiplicand and multiplying each term by $-b$. Thus:

(1)

$$\begin{array}{r} 5a\,(x+y) \\ -\,b \\ \hline -5ab\,(x+y) \end{array}\text{, result.}$$

(2)

$$\begin{array}{r} 5a\,(x+y) \\ -\,b \\ \hline 5a\,(-bx-by) \end{array}\text{, result.}$$

(3)

$$\begin{array}{r} 5a\,(x+y)=5ax+5ay \\ -b \\ \hline -5abx-5aby \end{array}\text{, result.}$$

12. $(a-b)^2=(a-b)(a-b)=(\quad)$?

Process:

$$\begin{array}{r} a-b \\ a-b \\ \hline -ab+b^2 \\ a^2-ab \\ \hline a^2-2ab+b^2 \end{array}\text{, result.}$$

PRINCIPLE: *1. The square of the difference of two numbers is equal to the square of the first minus twice the product of the first times the second plus the square of the second.* (Commit.)

NOTE.—Compare this principle with the one on page 181, noting the points of similarity and difference.

13. Multiply $a+b$ by $a-b$.

Process:

$$\begin{array}{r} a+b \\ a-b \\ \hline +ab-b^2 \\ a^2-ab \\ \hline a^2\qquad-b^2 \end{array}\text{, result.}$$

PRINCIPLE: *2. The product of the sum and difference of two numbers is equal to the square of the first minus the square of the second.* (Commit.)

EXERCISE CXXXIX.

1. Give *rule for addition.* Give principles.

2. Explain examples 1 to 4, letting the numbers represent distance,—the positive numbers distance measured to the right, and negative numbers distance measured to the left.

Thus, In Example 1,

$+8$, to the right.
$\underline{+4,}$ to the right.
$+12$; or, 12 to the right of starting-point.

In Example 3,

$+8$, to the right.
$\underline{-4,}$ to the left.
$+4$; or, still 4 to the right of starting-point.

Add (orally):

3.	*4.*	*5.*	*6.*	*7.*
-7	$+7$	$+7$	-7	14
-5	$+5$	-5	$+5$	-9

8.	*9.*	*10.*	*11.*	*12.*
-5	$+9$	-17	$+4$	-10
$+7$	-6	$+4$	-18	$+12$
-4	$+3$	-2	$+5$	-6

13.	*14.*	*15.*	*16.*	*17.*
$-a$	$-5x$	$+12xy$	$-3a^2x$	$-20x^2y^2$
$+5a$	$+25x$	$-15xy$	$+5a^2x$	$+18x^2y^2$
$-7a$	$-3x$	$+9xy$	$-12a^2x$	$+4x^2y^2$
$+a$	$-11x$	$-12xy$	$+4a^2x$	$-14x^2y^2$

18.	*19.*	*20.*
$7(x-2y)$	$-8(x^2-y)$	$9(x+y)^2$
$-9(x-2y)$	$+6(x^2-y)$	$-5(x+y)^2$
$+(x-2y)$	$-12(x^2-y)$	$+(x+y)^2$

Copy and add:

21. $12a-9c,\ -4a+17c,\ -7c+15a,\ 27a-c.$

22. $42a^2-5ab,\ -36a^2+12ab,\ 7a^2-15ab,\ -ab-14a^2.$

23. $a^2-2ab+b^2,\ 5a^2-9ab+4b^2,\ -12a^2+26ab-12b^2,\ 36a^2-24b^2+7ab.$

24. $11a^2b+11ab^2+a^2b^2,\ 5a^2b-21ab^2+6a^2b^2,\ 14ab^2-5a^2b^2+9a^2b,\ -4a^2b^2+5ab^2+3a^2b.$

25. $9x^2y+4x+3y-9y^2,\ 4y^2-2x^2y-5x+3y,\ 12x-14y-6y^2-2x^2y,\ x^2y-x+y-y^2.$

162. Subtraction.—It will help to explain the process of subtracting positive and negative numbers, if we will note the fact, *that any monomial may be expressed as a binomial, one of whose terms is positive and the other negative, and that the numerical value of either term may be increased at will, provided the numerical value of the other be correspondingly increased.*

EXAMPLES.

1. Represent $+5$ by an expression containing $+8$, $+12$, $+17$, -5, -14, -21.

(1) $5=8-3.$	(4) $5=10-5.$
(2) $5=12-7.$	(5) $5=19-14.$
(3) $5=17-12.$	(6) $5=26-21.$

2. Represent -7 by an expression containing $+4$, $+12$, $+15$, -9, -13, -25.

(1) $-7=-11+4.$	(4) $-7=-9+2.$
(2) $-7=-19+12.$	(5) $-7=-13+6.$
(3) $-7=-22+15.$	(6) $-7=-25+18.$

The process of subtraction consists in taking from the minuend the number expressed by the subtrahend.

EXAMPLES.

3. From $+8$ take $+4$.

(1) $+8-(+4)=+4$, result. Taking $+4$ from $+8$ leaves $+4$.

4. From -8 take $+4$.

(1) $-8=-12+4$.
(2) $(-12+4)-(+4)=-12$, result.

The -8 is the same as $-12+4$; taking away the $+4$ leaves -12.

5. From $+8$ take -4.

(1) $+8=+12-4$.
(2) $(+12-4)-(-4)=+12$, result.

The $+8$ is the same as $+12-4$; taking away the -4 leaves $+12$.

6. From -8 take -4.

(1) $-8-(-4)=-4$, result. Taking -4 from -8 leaves -4.

NOTE.—The same result will be obtained by the *capital* and *debt* plan of explanation. (See Addition, p. 336.)

Let us compare these examples:

(1) $+8-(+4)=+\ 4$.
(2) $-8-(+4)=-12$.
(3) $+8-(-4)=+12$.
(4) $-8-(-4)=-\ 4$.

Observe that, when the signs of the subtrahend and minuend were alike (in (1) and (4)), we took the difference between the 8 and the 4; when the signs of the subtrahend and minuend were unlike (in (2) and (3)), we took the sum of the 8 and the 4.

As we might have expected, this is exactly the reverse of our comparison in addition.

Suppose that in the four examples above, we change the sign of each subtrahend and add (instead of subtracting) and see what results we get:

(1) $+8+(-4)=+4$.
(2) $-8+(-4)=-12$.
(3) $+8+(+4)=+12$.
(4) $-8+(+4)=-4$.

These results are just the same as above, and we have subtracted by changing the signs of the subtrahends and adding.

RULE FOR SUBTRACTION: *Conceive the sign of the subtrahend to be changed, (the + to − or the − to +) and proceed as in addition.*

NOTE.—Do not actually change the sign of the subtrahend on paper

or slate in written work, but do it *mentally*. An actual change of written signs leads to confusion in reviewing the work.

PRINCIPLES: *1. The result can be expressed in one term only when the minuend and subtrahend are similar.*

2. The subtraction of dissimilar numbers is indicated by placing between them the proper sign.

7. From $12-5+4$ take $8-3+2$.

(1) $12-5+4=11$.	Collect the terms of both minuend and
(2) $8-3+2=7$.	subtrahend; change the sign of the sub-
(3) $11-7=4$.	trahend, and proceed as in addition. Or,
Or, $12-5+4-8+3-2=4$.	change all signs of the subtrahend, and then collect.

8. $5+17-3-(4+8-10+2)=?$

Process: $5+17-3-(4+8-10+2)=$
$5+17-3-4-8+10-2=15$, result.

NOTE.—The $-$ before the parentheses shows that all the numbers within are to be subtracted, and that the expression within the parentheses is a subtrahend.

RULE FOR CLEARING OF PARENTHESES PRECEDED BY A MINUS SIGN: *Drop the parentheses with the minus sign, and change the sign of every term within.*

9.	10.	11.	12
$7a$	$-xy$	$+3x^2$	$-17xy^2$
$5a$	$+5xy$	$-9x^2$	$-5xy^2$
$2a$, result.	$-6xy$, result.	$+12x^2$, result.	$-12xy^2$, result.

13. $7x^2+4ax+2b^2-(-5b^2+7xa-5x^2)=(\quad)$?

Process:
$$\begin{array}{r} 7x^2+4ax+2b^2 \\ -\ 5x^2+7ax-5b^2 \\ \hline 12x^2-3ax+7b^2 \end{array}$$ result.

Or,

$7x^2+4ax+2b^2-(-5b^2+7ax-5x^2)=$
$7x^2+4ax+2b^2+5b^2-7ax+5x^2=$
$12x^2-3ax+7b^2$, result.

14. From $5(a+x)$ take $2(a+x)$.

$$\begin{array}{l} 5(a+x) \\ \underline{2(a+x)} \\ 3(a+x), \text{ result.} \end{array}$$

EXERCISE CXL.

1. Explain examples 3 to 6, letting positive numbers mean *capital* and negative numbers mean *debt;* letting positive numbers mean distance moved to the *right*, and negative numbers distance moved to the *left.*

2. Give the *rule for subtraction.*

3. Give the *principles.*

4. Give the *rule for removing parentheses preceded by a minus sign.*

5. Do you change any signs in removing parentheses preceded by a plus sign?

Subtract (orally):

6.	*7.*	*8.*	*9.*	*10.*
-7	$+7$	$+7$	-7	12
-5	$+5$	-5	$+5$	-9

11.	*12.*	*13.*	*14.*	*15.*
$-a$	$-7a$	$-9a$	$-12x$	$+12x$
$-5a$	$+3a$	$-4a$	$-6x$	$+6x$

16.	*17.*	*18.*	*19.*	*20.*
$-a^2x$	$-9x^2y$	$-12mn$	$+ab$	$+17x^3$
$7a^2x$	$+12x^2y$	$-9mn$	$-20ab$	$-3x^3$

21.	*22.*	*23.*
$-9(x-2y)$	$+5(x^2-y^2)$	$7(x^2+2xy+y^2)$
$+4(x-2y)$	$+9(x^2-y^2)$	$-4(x^2+2xy+y^2)$

Copy and subtract:

24. From $3a-5b^2-c$ take $7a+4b^2-10c$.

25. From $7a^2+9xy-6b^3$ take $11b^3+4xy-10a^2$.

26. From $-75(x^2y-xy^2)$ take $-150(x^2y-xy^2)$.

27. $17x-5y+6-(5x+11y-10)=(\ \ )$?

28. $9xy-(7ax+12xy-3z)=(\ \ )$?

29. $7x^2y-3xy^2+y^3-(-7y^3+6xy^2-9yx^2)=(\ \ )$?

30. $a+b-c-d-x-(a-b+c-d-x)=(\ \ )$?

31. $x^2+2xy+y^2-(x^2-2xy+y^2)=(\ \ )$?

32. $x^3+3x^2y+3xy^2+y^3-(x^3-3x^2y+3xy^2-y^3)=(\ \ )$?

163. Multiplication.—When the multiplier is positive, it shows how many times the multiplicand is to be taken as an addend (see p. 21); but, when negative, it shows how many times the multiplicand is to be taken as a *subtrahend*.

EXAMPLES.

1. Multiply $+7$ by $+3$.

NOTE.—Starting with 0, we *add* $+7$ three times.

Process: (1) $0+(+7)=+7$, used once.
(2) $+7+(+7)=+14$, used twice.
(3) $+14+(+7)=+21$, used three times.
$\therefore\ +3\times+7=+21$, result.

2. Multiply -7 by $+3$.

NOTE.—Starting with 0, *add* -7 three times.

Process: (1) $0+(-7)=-7$, used once.
(2) $-7+(-7)=-14$, used twice.
(3) $-14+(-7)=-21$, used three times.
$\therefore\ +3\times-7=-21$, result.

3. Multiply $+7$ by -3.

NOTE.—Starting with 0, *subtract* $+7$ three times.

Process: (1) $0-(+7)=-7$, used once.
(2) $-7-(+7)=-14$, used twice.
(3) $-14-(+7)=-21$, used three times.
$\therefore\ -3\times+7=-21$, result.

4. Multiply -7 by -3.

NOTE.—Starting with 0, subtract -7 three times.

Process: (1) $0-(-7)=+7$, used once.
(2) $+7-(-7)=+14$, used twice.
(3) $+14-(-7)=+21$, used three times.
$\therefore\ -3\times-7=+21$.

Let us compare these examples:

(1) $+3\times+7=+21$.
(2) $+3\times-7=-21$.
(3) $-3\times+7=-21$.
(4) $-3\times-7=+21$.

Observe (1) that the numerical products are the same (each 21, or the product of 3 and 7), and (2) when the signs of the factors are alike (both plus as in (1) or both minus as in (4)), the product is *positive;* but when the signs are unlike (as in (2) and (3)), the product is *negative.*

LAW OF SIGNS IN MULTIPLICATION: *Like signs give plus; unlike signs give minus.*

5.	*6.*	*7.*	*8.*
$+12a$	$+12a$	$-12a$	$-12a$
$+6b$	$-6b$	$+6b$	$-6b$
$+72ab$, result.	$-72ab$, result.	$-72ab$, result.	$+72ab$, result.

9. $-7a\times+5b\times-3c=(\ \ )$?

Process: (1) $-7a\times+5b=-35ab$.
(2) $-35ab\times-3c=+105abc$, result.

10. Multiply $4m-7n$ by $5n$.

Process:
$$\begin{array}{r} 4m-7n \\ -5n \\ \hline -20mn+35n^2 \end{array}$$
$-20mn+35n^2$; or,
$35n^2-20mn$, result.

NOTE.—In writing polynomials, it is customary (but not necessary) to write a positive term (if there be one) on the left, and omit its sign.

11. Multiply $5a(x+y)$ by $-b$.

NOTE.—This multiplication may be performed (1) by multiplying the $5a$ by $-b$, (2) by multiplying $(x+y)$ by $-b$, or (3) by removing the parentheses from the multiplicand and multiplying each term by $-b$. Thus:

(1)

$$\begin{array}{r} 5a\,(x+y) \\ -b \\ \hline -5ab\,(x+y) \end{array}\text{, result.}$$

(2)

$$\begin{array}{r} 5a\,(x+y) \\ -b \\ \hline 5a\,(-bx-by) \end{array}\text{, result.}$$

(3) $5a\,(x+y) = 5ax+5ay$

$$\begin{array}{r} 5ax+5ay \\ -b \\ \hline -5abx-5aby \end{array}\text{, result.}$$

12. $(a-b)^2=(a-b)(a-b)=(\quad)$?

Process:

$$\begin{array}{r} a-b \\ a-b \\ \hline -ab+b^2 \\ a^2-ab \quad \\ \hline a^2-2ab+b^2 \end{array}\text{, result.}$$

PRINCIPLE: *1. The square of the difference of two numbers is equal to the square of the first minus twice the product of the first times the second plus the square of the second.* (Commit.)

NOTE.—Compare this principle with the one on page 181, noting the points of similarity and difference.

13. Multiply $a+b$ by $a-b$.

Process:

$$\begin{array}{r} a+b \\ a-b \\ \hline +ab-b^2 \\ a^2-ab \quad \\ \hline a^2 \quad\quad -b^2 \end{array}\text{, result.}$$

PRINCIPLE: *2. The product of the sum and difference of two numbers is equal to the square of the first minus the square of the second.* (Commit.)

14. Multiply $3a-2x$ by $5b-a$.

Process:

$$\begin{array}{r} 3a-2x \\ 5b\ \ -a \\ \hline 15ab-10bx-3a^2+2ax, \end{array}$$ result.

EXERCISE CXLI.

1. What change in the definition (p. 21) is necessary to make it include the multiplication of negative numbers?

2. What does a positive multiplier mean?

3. What does a negative multiplier mean? Give the *law of signs.*

4. Repeat principle 1. In what respect does it differ from that on p. 181?

5. Repeat principle 2.

Multiply (orally):

6. $-5\times+12$.	*10.* $-a\times+b$.	*14.* $(a+b)^2=(\ \ )$?
7. $+6\times-10$.	*11.* $-2a\times-3b$.	*15.* $(a-b)^2=(\ \ )$?
8. -7×-9.	*12.* $+5a\times+7m^2$.	*16.* $(x-y)^2=(\ \ )$?
9. $-8\times+11$.	*13.* $-4x^2\times12y^2$.	*17.* $(x-y)(x+y)=(\ \ )$?

Copy and multiply:

18. $-9(x-y)$ by $-5a$.

19. $7x-12y$ by $a-b$.

20. $24m^2-7mn$ by $6m-11n$.

21. $36xy-5xy^2$ by $3x-5z$.

22. $x^2+2xy+y^2$ by $x+y$.

23. $x^3-3x^2y+3xy^2-y^2$ by x^2-y^2.

24. $5xy-7am+n^2$ by $6a-7b$.

164. Division.—By principle 3 in division (page 25), the dividend equals the product of the divisor and quotient.

Since (1) $+7\times+3=+21$, $+21\div+7=+3$.
Since (2) $-7\times+3=-21$, $-21\div-7=+3$.
Since (3) $-7\times-3=+21$, $+21\div-7=-3$.
Since (4) $+7\times-3=-21$, $-21\div+7=-3$.

Observe that when the signs of the dividend and divisor are alike the quotient is *positive*, and when the signs of the dividend and divisor are unlike the quotient is *negative*.

LAW OF SIGNS IN DIVISION: *Like signs give plus; unlike signs give minus.*

EXAMPLES.

1. $6)\underline{24}$
4, result.

2. $-6)\underline{-24}$
4, result.

3. $-6)\underline{24}$
-4, result.

4. $6)\underline{-24}$
-4, result.

5. $4y)\underline{24xy}$
$6x$, result.

6. $-6x)\underline{-24xy}$
$4y$, result.

7. $+6y)\underline{-24xy}$
$-4x$, result.

8. $-4y)\underline{24xy}$
$-6x$, result.

9. $6y)\underline{12xy-6y^2}$
$2x\ -y$, result.

10. $-5x(\underline{15ax-10xy}$
$-3a+2y$; or,
$2y-3a$, result.

11. Divide $x^2-2xy+y^2$ by $x-y$.

Process: $x-y)\ x^2-2xy+y^2(x-y$, result.

$$\begin{array}{l}\underline{x^2\ -xy}\\ \quad -xy+y^2\\ \quad \underline{-xy+y^2}\end{array}$$

EXPLANATION: (1) Find how many times the first term of the dividend contains the first term of the divisor. $x^2\div x=x$. Place x in the quotient. Multiply the whole divisor by the quotient x, and subtract the product from the dividend.

(2) *Proceed just as before.* Find how many times the first term of the remaining dividend contains the first term of the divisor. $-xy\div x=-y$. Place $-y$ in the quotient. Multiply the whole divisor by $-y$ and subtract. Result, $x-y$.

12. Divide x^2-y^2 by $x+y$.

Process: $x+y)\ x^2-y^2\ (x-y$, result.

$$\begin{array}{l} x^2+xy \\ \hline -xy-y^2 \\ -xy-y^2 \end{array}$$

NOTE.—When we go to subtract $+xy$, there is no similar term in the dividend; so the remainder is $-xy$, to which we bring down the next term of the dividend, making $-xy-y^2$.

EXERCISE CXLII.

1. What is the *law of signs* in division? Compare with the law of signs in multiplication.

Divide (orally):

2. $25\div5$.

3. $-25\div5$.

4. $25\div-5$.

5. $-25\div-5$.

6. $-10\,ab\div+ab$.

7. $-25\,a^2x\div-5\,ax$.

8. $-36x^2y^2\div9\,xy$.

9. $17\,ax^2y\div axy$.

10. $-12\,ax(m+n)\div12\,ax$.

11. $-12\,ax\,(y-z)\div(y-z)$.

Copy and divide—

12. $5\,m^2n-4\,mn^2$ by mn.

13. $15\,x^2-25\,xy^2$ by $5\,x$.

14. $x^2+2\,xy+y^2$ by $x+y$.

15. $x^2-2\,ax+a^2$ by $x-a$.

16. $x^3-3\,x^2a+3\,xa^2-a^3$ by $x^2-2\,ax+a^2$.

17. $x^3+3\,x^2y+3\,xy^2+y^3$ by $x+y$.

165. Factoring.—The factors of numbers may be found by *inspection* or by *division*. (See pp. 40 and 41.)

EXAMPLES.

1. Factor $5\,ab$.

Process: $5\,ab=5\times a\times b$.

Factors, 5, a, and b.

2. Factor $25a^2c$ (Inspection).

Factors, 5, 5, a, a, and c.

3. Factor $5b^2(x+y)$.

Factors, 5, b, b, and $x+y$.

4. Factor $5ab-10b^2$.

Process: $5b\,)\,\underline{5ab-10b^2}$
$\quad\quad\quad\quad a \;-\; 2b.$

Factors, 5, b, and $a-2b$.

5. Factor $a^2+2ab+b^2$.

NOTE.—By the principle (p. 181), $a^2+2ab+b^2$ is the square of the sum of two numbers.

Process: $a^2+2ab+b^2=(a+b)\,(a+b)$.

Factors, $a+b$ and $a+b$.

6. Factor $a^2-2ab+b^2$.

NOTE.—Apply prin. 1, p. 345.

Factors, $a-b$ and $a-b$.

7. Factor a^2-b^2.

NOTE.—Apply prin. 2, p. 345.

Factors, $a+b$ and $a-b$.

EXERCISE CXLIII.

Factor—

1. $12a$.	*5.* $48x^2y^2$.	*9.* $12(x-y)$.
2. $15b^2$.	*6.* $51mn^3$.	*10.* $x(m+n)$.
3. $25ab$.	*7.* $64x^3z^2$.	*11.* $16xy(a+b)$.
4. $72m^2n$.	*8.* $121x^4y^2z$.	*12.* $75x^2(a+b)^2$

13. $m^2+2mn+n^2$.	*16.* x^2-2x+1.
14. $x^2-2xz+z^2$.	*17.* x^2-y^2.
15. a^2+2a+1.	*18.* $x^3-3x^2y+3xy^2-y^3$.

166. Greatest Common Divisor.—The G. C. D is usually found by factoring. The plan here is not different from that found on page 47.

EXAMPLES.

1. Find the G. C. D. of $5ac$, $25ax$, $15an$.

Process: (1) $5ac = 5 \times a \times c$.
(2) $25ax = 5 \times 5 \times a \times x$.
(3) $15an = 3 \times 5 \times a \times n$.
$\therefore$ G. C. D. $= 5 \times a = 5a$.

2. Find the G. C. D. of $5a(x-y)$, $15mx-15my$.

Process: (1) $5a(x-y) = 5 \times a \times (x-y)$.
(2) $15mx-15my = 3 \times 5 \times m \times (x-y)$.
$\therefore$ G. C. D. $= 5 \times (x-y) = 5(x-y)$; or, $5x-5y$.

3. Find the G. C. D. of $x^2+2xy+y^2$, x^2-y^2, $7x+7y$.

Process: (1) $x^2+2xy+y^2 = (x+y)(x+y)$.
(2) $x^2-y^2 = (x+y)(x-y)$.
(3) $7x+7y = 7(x+y)$.
$\therefore$ G. C. D. $= x+y$.

EXERCISE CXLIV.

Find the G. C. D. of—

1. a^2, $5ax$, $3ay$.
2. $7mx$, $14am$, $35my$.
3. $24x^2y^2$, $16xy^2$, $40x^2yz^3$.
4. $12a^2m^2n^2$, $18am^3n$, $36a^2m^2n^2$.
5. $3ax(m-n)$, m^2-n^2, $7m-7n$.
6. $25ax(y-z)$, $15ay^2-15az^2$, $10ay-10az$.
7. $x^2+2xy+y^2$, x^2-y^2, $17ax+17ay$.
8. $(x-y)^2$, x^2-y^2, $5bx-5by$.

167. Least Common Multiple.—The L. C. M. is usually found by factoring. The plan here is not different from that found on page 49.

EXAMPLES.

1. Find the L. C. M. of $5a$, $10ay$, $15a^2y^2$.

Process: (1) $5a = 5\times a$.
(2) $10ay = 2\times 5\times a\times y$.
(3) $15a^2y^2 = 3\times 5\times a\times a\times y\times y$.
$\therefore$ L. C. M. $= 2\times 3\times 5\times a\times a\times y\times y = 30a^2y^2$.

2. Find the L. C. M. of $5a(x-y)$, $7ab$, $35a(x-y)^2$.

Process: (1) $5a(x-y) = 5\times a\times(x-y)$.
(2) $7ab = 7\times a\times b$.
(3) $35a(x-y)^2 = 5\times 7\times a\times(x-y)(x-y)$.
$\therefore$ L. C. M. $= 5\times 7\times a\times b\times(x-y)(x-y) = 35ab(x-y)^2$; or, $35ab(x^2-2xy+y^2)$.

EXERCISE CXLV.

Find the L. C. M. of—

1. $7a$, $14ab$, $28a^2b$.
2. $9y^2$, $18xy$, $24x^2y^2$.
3. $2mn^2z$, $6m^2nz$, $12mnz^2$.
4. ms^2, $5mns$, $20xns$.
5. $15ax(m-n)$, 25, $10a(m-n)$, $5x(m^2-n^2)$.
6. 12, x^2-y^2, $(x-y)^2$, $6(x-y)$.
7. $8b$, $12c$, $3b^2c$, $4b(m-n)$.
8. $x^2-2xy+y^2$, $x^2+2xy+y^2$, x^2-y^2.

B. FRACTIONAL NUMBERS.

168. Reduction.—The reduction of fractions here does not differ in principle from that found in Articles 35 to 39.

EXAMPLES.

1. Reduce $\frac{5a}{2b}$ to a fraction whose denominator is $10bx$.

Process: (1) $10bx \div 2b = 5x$.
(2) $\frac{5x\times 5a}{5x\times 2b} = \frac{25ax}{10bx}$, result.

2. Reduce $\frac{48\,ax^2y}{72\,a^2x(m-n)}$ to its lowest terms.

NOTE.—24 ax is the G. C. D. of the numerator and the denominator.

$$\textit{Process:}\quad \frac{48\,ax^2y \div 24\,ax}{72\,a^2x(m-n) \div 24\,ax} = \frac{2\,xy}{3\,a(m-n)},\ \text{result.}$$

3. Reduce $5x-a-\frac{ab}{xy}$ to the form of a fraction.

$$\textit{Process:}\quad 5x-a-\frac{ab}{xy} = \frac{xy\,(5x-a)-ab}{xy} = \frac{5x^2y-axy-ab}{xy},\ \text{result.}$$

NOTE.—(1) Multiply the integral part by the denominator, (2) add the numerator, and (3) place the sum over the denominator.

4. Reduce $\frac{a^2-2\,ab+3\,b^2}{a-b}$ to a mixed number.

$$\textit{Process:}\quad a-b\,)\,a^2-2\,ab+3\,b^2\,(\,a-b+\frac{2\,b^2}{a-b},\ \text{result.}$$
$$\begin{array}{r} a^2 - ab \\ \hline -ab+3\,b^2 \\ -ab\ \ +b^2 \\ \hline +2\,b^2,\ \text{remainder.} \end{array}$$

5. $\frac{5}{a}, \frac{2\,b}{6\,c}, \frac{5\,m}{a^2d}$ to least common denominator.

NOTE.—L. C. D. of a, $6\,c$, and a^2d is $6\,a^2cd$.

$$\textit{Process:}\quad (1)\ 6\,a^2cd \div a = 6\,acd;\ \frac{6\,acd \times 5}{6\,acd \times a} = \frac{30\,acd}{6\,a^2cd}.$$
$$(2)\ 6\,a^2cd \div 6\,c = a^2d;\ \frac{a^2d \times 2\,b}{a^2d \times 6\,c} = \frac{2\,a^2bd}{6\,a^2cd}.$$
$$(3)\ 6\,a^2cd \div a^2d = 6\,c;\ \frac{6\,c \times 5\,m}{6\,c \times a^2d} = \frac{30\,cm}{6\,a^2cd}.$$

EXERCISE CXLVI.

Reduce—

1. 5 to a fraction whose denominator is x.

2. $\frac{3\,a}{mz}$ to a fraction whose denominator is $7\,amz$.

3. $\frac{25axy^2}{50a^2my^3}$ to lowest terms.

4. $\frac{12xy(m-n)}{18x^2(m^2-n^2)}$ to lowest terms.

5. $\frac{x^2+2xy+y^2}{x^2-y^2}$ to lowest terms.

6. $\frac{x^2-5y^2}{x+y}$ to mixed number.

7. $\frac{x^2+2xy-2y^2}{x+y}$ to mixed number.

8. $a^2-\frac{b}{c}$ to fractional form.

9. $b^2y-m+\frac{c}{x^2y}$ to fractional form.

10. $\frac{a}{b}, \frac{2b}{ac}, \frac{3x}{ab}$ to L. C. D.

11. $7, \frac{a}{x}, \frac{y^2}{b}, \frac{a}{b}$ to L. C. D.

12. $\frac{5x}{a^2y}, \frac{7b}{ay^2}, \frac{7z}{ay(m-n)}$ to L. C. D.

169. Addition.—Addition of fractions here does not differ in principle from that found in Article 40.

EXAMPLES.

1. Add $\frac{3x}{4}, \frac{5x}{8}, \frac{7x}{12}$.

Process: $\frac{3x}{4}+\frac{5x}{8}+\frac{7x}{12}=\frac{18x+15x+14x}{24}=\frac{47}{24}x$, result.

Note: $\frac{47}{24}x$ is the same as $\frac{47x}{24}$.

2. Add $\frac{a}{b}, \frac{c}{ab}, \frac{c}{5b}$.

Process: $\frac{a}{b}+\frac{c}{ab}+\frac{c}{5b}=\frac{5a^2+5c+ac}{5ab}$, result.

3. Add $2b$, $+\frac{7a}{x}$, $-\frac{8bm}{m-n}$.

Process: $2b+\frac{7a}{x}-\frac{8bm}{m-n}=$

$$\frac{2bmx-2bnx}{xm-xn}+\frac{7am-7an}{xm-xn}-\frac{8bmx}{xm-xn}=$$

$$\frac{2bmx-2bnx+7am-7an-8bmx}{xm-xn}=$$

$$\frac{7am-2bnx-7an-6bmx}{xm-xn}\text{, result.}$$

NOTE.—Have the pupil explain each step.

EXERCISE CXLVII.

Add:

1. $\frac{a}{6}$, $+\frac{a}{4}$, $+\frac{a}{3}$.

2. $\frac{2a}{x}$, $-\frac{3}{y}$, $+\frac{7}{xy}$.

3. $\frac{x-y}{2}$, $\frac{x+y}{2}$.

4. $\frac{mn}{5}$, $-\frac{3mn}{6}$, $+\frac{5mn}{10}$.

5. $\frac{5x-5y}{8}$, $-\frac{7x-3y}{12}$.

6. $\frac{5-x}{a}$, b, $-\frac{3x-b}{d}$.

7. $\frac{5}{3x+3y}$, $-\frac{3}{5x+5y}$, $\frac{x-y}{(x+y)^2}$.

8. $\frac{5a}{mc}+\frac{7ab}{5m}-\frac{3a}{4c}=(\quad)$?

9. $3x-\frac{7y}{2z}+\frac{9b^2z}{a-b}=(\quad)$?

170. Subtraction.—Subtraction of fractions here does not differ in principle from that found in Article 41.

EXAMPLES.

1. From $\frac{x}{4}$ take $\frac{x}{5}$.

Process: $\frac{x}{4}-\frac{x}{5}=\frac{5x}{20}-\frac{4x}{20}=\frac{x}{20}$, result.

2. From $\frac{5y}{7}$ take $-\frac{4y}{9}$.

Process: (1) $\frac{5y}{7}-\left[-\frac{4y}{9}\right]=\frac{5y}{7}+\frac{4y}{9}$. (Why?)

(2) $\frac{5y}{7}+\frac{4y}{9}=\frac{45y+28y}{63}=\frac{73y}{63}$, result.

3. $\frac{5x}{a}-\left[\frac{5x+3y}{6a}\right]=(\quad)$?

Process: $\frac{5x}{a}-\left[\frac{5x+3y}{6a}\right]=\frac{30x}{6a}-\left[\frac{5x+3y}{6a}\right]=$

$\frac{30x-(5x+3y)}{6a}=\frac{25x-3y}{6a}$, result.

4. From $\frac{6x-5}{x-y}$ take $\frac{6x-5}{x+y}$.

Process: $\frac{6x-5}{x-y}-\frac{6x-5}{x+y}=\frac{6x^2-5x+6xy-5y}{x^2-y^2}-\frac{6x^2-5x-6xy+5y}{x^2-y^2}=$

$\frac{6x^2-5x+6xy-5y-(6x^2-5x-6xy+5y)}{x^2-y^2}=$

$\frac{6x^2-5x+6xy-5y-6x^2+5x+6xy-5y}{x^2-y^2}=\frac{12xy-10y}{x^2-y^2}$, result.

EXERCISE CXLVIII.

1. From $\frac{7x}{5}$ take $\frac{5x}{9}$.

2. From $\frac{12y}{x}$ take $\frac{5y}{11x}$.

3. From $\frac{7}{y}$ take $\frac{10}{13y}$.

4. From $\frac{x^2}{y}$ take $\frac{x^2}{7y}$.

5. From $\frac{a+b}{12}$ take $\frac{a-b}{15}$.

6. From $\frac{m}{x+y}$ take $\frac{m}{x-y}$.

3. Add $2b$, $+\frac{7a}{x}$, $-\frac{8bm}{m-n}$.

Process: $2b+\frac{7a}{x}-\frac{8bm}{m-n}=$

$$\frac{2bmx-2bnx}{xm-xn}+\frac{7am-7an}{xm-xn}-\frac{8bmx}{xm-xn}=$$

$$\frac{2bmx-2bnx+7am-7an-8bmx}{xm-xn}=$$

$$\frac{7am-2bnx-7an-6bmx}{xm-xn}, \text{ result.}$$

NOTE.—Have the pupil explain each step.

EXERCISE CXLVII.

Add:

1. $\frac{a}{6}$, $+\frac{a}{4}$, $+\frac{a}{3}$.

2. $\frac{2a}{x}$, $-\frac{3}{y}$, $+\frac{7}{xy}$.

3. $\frac{x-y}{2}$, $\frac{x+y}{2}$.

4. $\frac{mn}{5}$, $-\frac{3mn}{6}$, $+\frac{5mn}{10}$.

5. $\frac{5x-5y}{8}$, $-\frac{7x-3y}{12}$.

6. $\frac{5-x}{a}$, b, $-\frac{3x-b}{d}$.

7. $\frac{5}{3x+3y}$, $-\frac{3}{5x+5y}$, $\frac{x-y}{(x+y)^2}$.

8. $\frac{5a}{mc}+\frac{7ab}{5m}-\frac{3a}{4c}=(\quad)$?

9. $3x-\frac{7y}{2z}+\frac{9b^2z}{a-b}=(\quad)$?

170. Subtraction.—Subtraction of fractions here does not differ in principle from that found in Article 41.

EXAMPLES.

1. From $\frac{x}{4}$ take $\frac{x}{5}$.

Process: $\frac{x}{4}-\frac{x}{5}=\frac{5x}{20}-\frac{4x}{20}=\frac{x}{20}$, result.

2. From $\frac{5y}{7}$ take $-\frac{4y}{9}$.

Process: (1) $\frac{5y}{7}-\left[-\frac{4y}{9}\right]=\frac{5y}{7}+\frac{4y}{9}$. (Why?)

(2) $\frac{5y}{7}+\frac{4y}{9}=\frac{45y+28y}{63}=\frac{73y}{63}$, result.

3. $\frac{5x}{a}-\left[\frac{5x+3y}{6a}\right]=(\ \ )$?

Process: $\frac{5x}{a}-\left[\frac{5x+3y}{6a}\right]=\frac{30x}{6a}-\left[\frac{5x+3y}{6a}\right]=$

$\frac{30x-(5x+3y)}{6a}=\frac{25x-3y}{6a}$, result.

4. From $\frac{6x-5}{x-y}$ take $\frac{6x-5}{x+y}$.

Process: $\frac{6x-5}{x-y}-\frac{6x-5}{x+y}=\frac{6x^2-5x+6xy-5y}{x^2-y^2}-\frac{6x^2-5x-6xy+5y}{x^2-y^2}=$

$\frac{6x^2-5x+6xy-5y-(6x^2-5x-6xy+5y)}{x^2-y^2}=$

$\frac{6x^2-5x+6xy-5y-6x^2+5x+6xy-5y}{x^2-y^2}=\frac{12xy-10y}{x^2-y^2}$, result.

EXERCISE CXLVIII.

1. From $\frac{7x}{5}$ take $\frac{5x}{9}$.

2. From $\frac{12y}{x}$ take $\frac{5y}{11x}$.

3. From $\frac{7}{y}$ take $\frac{10}{13y}$.

4. From $\frac{x^2}{y}$ take $\frac{x^2}{7y}$.

5. From $\frac{a+b}{12}$ take $\frac{a-b}{15}$.

6. From $\frac{m}{x+y}$ take $\frac{m}{x-y}$.

7. From $\frac{12x-10}{x+2y}$ take $\frac{6x-5}{3x+6y}$.

8. $a-\frac{3b}{10}-\left\{\frac{7a-5b}{9}-b\right\}=(\quad)$?

171. Multiplication.—Multiplication of fractions here does not differ in principle from that found in Article 42.

EXAMPLES.

1. Multiply $\frac{5a}{6}$ by $\frac{3b}{ax}$.

Process: $\frac{3b}{ax}\times\frac{5a}{6}=\frac{5b}{2x}$, result.

NOTE.—Cancel, when possible.

2. $\frac{a-b}{a+b}\times\frac{m+n}{a-b}=(\quad)$?

Process: $\frac{a-b}{a+b}\times\frac{m+n}{a-b}=\frac{m+n}{a+b}$, result.

3. $\frac{a}{b}\times-\frac{c}{b}\times\frac{a}{3c}=(\quad)$?

Process: $\frac{a}{b}\times-\frac{c}{b}\times\frac{a}{3c}=-\frac{a^2}{3b^2}$, result.

EXERCISE CXLIX.

Multiply—

1. $\frac{5}{b}$ by $\frac{3x}{10c}$.
2. $\frac{7x}{3y}$ by $\frac{6y}{12w}$.
3. $\frac{5ab}{7xy}$ by $\frac{14xz}{10a^2b}$.
4. $3ab$ by $\frac{5m}{7ax}$.
5. $\frac{3axy}{22b^3c}$ by $\frac{11b^2y}{12ax^2}$.
6. $\frac{x+y}{x+z}$ by $\frac{x-y}{x-z}$.
7. $\frac{x(a+b)}{a-b}$ by $\frac{a-b}{12(x+a)}$.
8. $\frac{5a-6b}{3x+6y}$ by $\frac{5m-3n}{5n-3b}$.

172. Division.—Division of fractions here does not differ in principle from that found in Article 43.

EXAMPLES.

1. Divide $\frac{5a}{m}$ by $\frac{3b}{m^2}$.

Process: $\frac{5a}{m} \div \frac{3b}{m^2} = \frac{5a}{m} \times \frac{m^2}{3b} = \frac{5am}{3b}$, result.

2. Divide $\frac{m-n}{m+n}$ by $\frac{m-n}{x+y}$.

Process: $\frac{m-n}{m+n} \div \frac{m-n}{x+y} = \frac{m-n}{m+n} \times \frac{x+y}{m-n} = \frac{x+y}{m+n}$, result.

3. Divide $\frac{3a}{4m}$ by $5b$.

Process: $\frac{3a}{4m} \div 5b = \frac{3a}{4m} \times \frac{1}{5b} = \frac{3a}{20bm}$, result.

EXERCISE CL.

Divide—

1. $\frac{7x}{5}$ by $\frac{9z}{10}$.

2. $\frac{81xy}{25z}$ by $\frac{90x^2}{35z^2}$.

3. $\frac{14z^2}{27ay}$ by $\frac{5zx}{9my}$.

4. $\frac{5abc}{7xyz}$ by $\frac{20abc^2}{12x^2m}$.

5. $\frac{8z-4}{7x}$ by $\frac{4z-2}{5}$.

6. $\frac{x+y}{m+n}$ by $\frac{a-b}{2m+2n}$.

7. $\dfrac{\frac{12-9x}{4}}{\frac{16-12x}{x+y}} = (\quad)$?

II. STUDY OF PROBLEMS.

A. PROBLEMS OF ONE BASIS.

173. Solving Equations of One Unknown Number.—We have learned that various operations may be performed on an equation without destroying the equality:

(1) *The terms may be transposed.* (See p. 81.)

EXAMPLES.

1. $5x-15=4x-5$. Find the value of x.

Process : (1) $5x-15=4x-5$.
Transposing, (2) $5x-4x=15-5$.
(2) = (3) $x=10$, **result.**

(2) *An equation may be multiplied.* (See p. 83.)

2. $\frac{x}{4}-\frac{x}{5}=7$. Find the value of x.

Process : (1) $\frac{x}{4}-\frac{x}{5}=7$.
(1) = (2) $\frac{5x-4x}{20}=7$.
$20\times(2)=(3)$ $5x-4x=140$.
(3) = (4) $x=140$, **result.**

PRINCIPLES: *1. Any equation may be cleared of fractions by being multiplied by the L. C. M. of the denominators of all the fractions.*

3. $\frac{7x}{8}-\frac{6x+4}{7}=13$. Find the value of x.

Process : (1) $\frac{7x}{8}-\frac{6x+4}{7}=13$.
$56\times(1)=(2)$ $49x-(48x+32)=728$.
(2) = (3) $49x-(48x-32=728$.
Transposing, (4) $49x-48x=728+32$.
Collecting, (5) $x=760$, **result.**

2. The signs of all the terms of an equation may be changed by multiplying the equation by -1.

4. $5x+3=6x-9$. Find the value of x.

Process : (1) $5x+3=6x-9$.
Trans., (2) $5x-6x=-9-3$.
Col., (4) $-x=-12$.
$-1\times(4)=(5)$ $x=12$, **result.**

NOTE.—In practice, we simply rewrite the equation with the signs changed.

(*3*) *An equation may be divided.* (See p. 89.)

5. $35x-15=18x+70$. Find the value of x.

Process: (1) $35x-15=18x+70$.
Trans., (2) $35x-18x=70+15$.
Col., (3) $17x=85$.
(3) ÷ 17 = (4) $x=5$, result.

PLAN: In finding the value of the unknown number in an equation, it is best to follow a definite plan. The operations required should usually be performed in the following order:

(1) Clear of fractions.

(2) Clear of parentheses.

(3) Transpose all terms containing the unknown number to the left member of the equation, and all other terms to the right member.

(4) Collect the terms of each member into one term.

(5) Change the signs.*

(6) Divide by the coefficient of the unknown number.

NOTE.—It is not often that all these operations will be required in solving one equation, but it is usually best to perform those that are required in the order indicated.

EXERCISE CLI.

1. What is an *equation*?

2. What principles are involved in the transposition of the terms of an equation (page 81)? Give the *law* of transposition (page 82).

3. What is the rule for removing parentheses (page 341)?

4. Give principle for clearing of fractions.

5. Upon what principle may the signs of all the terms of an equation be changed?

* The (5) may be dispensed with, by dividing by the negative coefficient in the (6).

Find the value of the unknown number in—

6. $7x-15=5x+9.$
7. $25-5x=0.$
8. $7y+12=43y.$
9. $5x-18-\frac{3}{4}x-33=0.$
10. $9x+4x-\frac{3}{8}x=25\frac{1}{4}.$
11. $5n+16+3n=10n.$
12. $-z+\frac{1}{3}z=51-63.$
13. $24a-12\frac{1}{2}-6\frac{1}{4}a=5\frac{1}{4}.$
14. $\frac{5x}{11}=15.$
15. $\frac{3b}{7}=21.$
16. $\frac{w}{4}-\frac{w}{6}=3.$
17. $\frac{m}{4}+\frac{m}{8}=26.$
18. $\frac{x}{5}+\frac{x}{6}+\frac{x}{3}=42.$
19. $\frac{5x}{6}-\frac{3x}{11}=111.$
20. $12x-\frac{5x}{7}-\frac{3x}{4}=42\frac{1}{7}.$
21. $7x+30-\frac{3x}{4}=205.$
22. $\frac{x-1}{5}+\frac{3x}{4}=1\frac{7}{10}.$
23. $\frac{7x}{3}+\frac{2x-4}{5}=10\frac{2}{15}.$
24. $\frac{3y-7}{9}-\frac{4y-2}{3}=-7\frac{1}{3}.$
25. $\frac{5m+5}{4}-\frac{6m-7}{3}=\frac{9m-5\frac{1}{2}}{6}.$

174. Problems.

EXAMPLES.

1. $\frac{1}{3}$ of a number, $\frac{1}{4}$ of the number, and 10, are together equal to the number. Find the number.

Solution: Let x = the number.*

Then, (1) $\frac{1}{3}x+\frac{1}{4}x+10=x$. (Why?)

$12\times(1)=(2)$ $4x+3x+120=12x.$

$(2)=(3)$ $12x=120+4x+3x.$

Trans., (4) $12x-4x-3x=120.$

Col., (5) $5x=120.$

$\frac{1}{5}$ of $(5)=(6)$ $x=24$, answer.

2. Find a number whose $\frac{1}{3}$ exceeds its $\frac{1}{4}$ by 150.

Solution: Let x = the number.

Then, (1) $\frac{1}{3}x-\frac{1}{4}x=150$. (Why?)

$12\times(1)=(2)$ $4x-3x=1800.$

Col., (3) $x=1800$, answer.

* "Let x=the number," means that x is to be used *to represent* the number. The sign = is used because it is more convenient than the word *represent*.

3. The sum of two numbers is 25, and their difference is 5. Find the numbers.

Solution: Let $x=$ the smaller number.
Then, $25-x=$ the larger number
and (1) $25-x-x=5$. (Why?)
Trans., (2) $-x-x=5-25$.
Col., (3) $-2x=-20$.
(3) $\div -2=$ (4) $x=10$, smaller number.
(5) $25-x=15$, larger number.

4. What four successive integers are together equal to 46?

Solution: Let $x=$ 1st number.
Then, $x+1=$ 2d number,
$x+2=$ 3d number,
$x+3=$ 4th number,
and (1) $x+(x+1)+(x+2)+(x+3)=46$. (Why?)
(1) = (2) $x+x+1+x+2+x+3=46$.
Trans., (3) $x+x+x+x=46-1-2-3$.
Col., (4) $4x=40$.
$\frac{1}{4}$ of (4) = (5) $x=10$, 1st number.
Then, the other numbers are 11, 12, and 13.

5. Henry spent $\frac{2}{11}$ of his money, and then received \$65; he then lost $\frac{3}{4}$ of all his money, and had in hand \$10 less than at first. How much had he at first?

Solution: Let $x=$ his money at first.
Then, $\frac{9}{11}x=$ amt. left,
$\frac{9}{11}x+65=$ amt. after receiving \$65,
and (1) $\frac{1}{4}(\frac{9}{11}x+65)=x-10$. (Why?)
(1) = (2) $\frac{9}{44}x+\frac{65}{4}=x-10$.
$44\times$(2) = (3) $9x+715=44x-440$.
Trans., (4) $9x-44x=-440-715$.
Col., (5) $-35x=-1155$.
(5) $\div -35=$ (6) $x=33$.
Answer, \$33.

6. Goods sold at 10% gain; if the goods had cost \$120 more, the loss would have been 10%. Find the cost price.

NOTE.—The selling price is 110% or $\frac{11}{10}$ of the *real cost*, and 90% or $\frac{9}{10}$ of the *supposed cost*.

Solution: Let x = real cost.

Then, $x+120$ = supposed cost

and (1) $\frac{11}{10}x=\frac{9}{10}(x+120)$. (Why?)

$10\times(1)=(2)$ $11x=9x+1080$.

Trans., (3) $11x-9x=1080$.

Col., (4) $2x=1080$.

$\frac{1}{2}$ of $(4)=(5)$ $x=540$.

Answer, \$540.

7. By selling my watch for \$36, I lose $\frac{2}{5}$ of its cost. What did it cost?

Equation: $x-\frac{2}{5}x=36$. (Why?)

8. A and B have equal sums of money; A gains \$100 and B loses \$150; then, twice A's money is equal to three times B's. What sum had each at first?

Equation: $2(x+100)=3(x-150)$. (Why?)

9. In a certain weight of gunpowder the saltpetre is 6 lb. more than $\frac{1}{2}$ of the whole, the sulphur is 5 lb. less than $\frac{1}{3}$ of the whole, and the charcoal is 3 lb. less than $\frac{1}{4}$ of the whole. Find the entire weight and the weight of each part.

Equation: Let x = whole weight.

Then, (1) $(\frac{1}{2}x+6)+(\frac{1}{3}x-5)+(\frac{1}{4}x-3)=x$. (Why?)

10. 6 times the ratio of $\frac{1}{3}$ to $\frac{3}{4}$ is equal to how many times the ratio of 3 to 9?

Solution: Let x = required number.

Then, (1) $6\times(\frac{1}{3}:\frac{3}{4})=x\times(3:9)$. (Why?)

$(1)=(2)$ $6\times\frac{4}{9}=x\times\frac{1}{3}$.

$(2)=(3)$ $\frac{1}{3}x=\frac{8}{3}$.

$3\times(3)=(4)$ $x=8$, answer.

11. A and B can perform a piece of work in 10 days. They both work three days and B then finishes it in 12 days. How long will it take each to do the work?

Equation : Let x = time reqd. by A to do the work.

Then, $\frac{1}{x}$ = part done by A in 1 da.,

$\frac{1}{10}$ = part done by both in 1 da.,

$\frac{3}{10}$ = part done by both in 3 da.,

$\frac{1}{10}-\frac{1}{x}$ = part done by B in 1 da.,

$\frac{12}{10}-\frac{12}{x}$ = part done by B in 12 da.,

and (1) $\frac{3}{10}+\frac{12}{10}-\frac{12}{x}=1.$ (Why?)

NOTE.—The question is sometimes asked, "Where did you get the *1?*" The concrete form of the equation will explain:

$$\frac{3}{10}\text{ of work}+\frac{12}{10}\text{ of work}-\frac{12}{x}\text{ of work}=1\times\text{the work.}$$

12. If I subtract \$20 from $\frac{1}{5}$ of my money, multiply the difference by 7, and subtract this product from \$1300, the result will be equal to my money. How much have I?

Equation : $1300-7(\frac{1}{5}x-20)=x.$ (Why?)

NOTE.—Always require the pupil to explain his reasons for forming the equation; because there is where the pupil does his investigative thinking. Solving the equation is *mechanical.*

EXERCISE CLII.

1. $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ of a number exceed the number by 14: find the number.

2. A number decreased by its $\frac{1}{5}$ and $\frac{2}{7}$ equals 54: find the number.

3. A number increased by its $\frac{1}{8}$ and $\frac{3}{10}$ equals 171: find the number.

4. $\frac{4}{5}$ of a number is 30 greater than its $\frac{1}{2}$: find the number.

5. 5 times a certain number is 27 more than its $\frac{1}{2}$: find the number.

6. The sum of 3 successive numbers is 72: find the numbers.

7. The sum of 5 numbers each differing from the next smaller by 4 is 140: find the numbers.

8. The sum of 3 numbers each 3 times the next smaller is 39: find the numbers.

9. A and B had the same amount of money; A lost \$40 and B gained \$60; $\frac{1}{4}$ of A's money was then equal to $\frac{1}{5}$ of B's: what did each have at the start?

10. A and B had the same amount of money; A gave B \$10; B then had 5 times as much as A: what sum had each at first?

11. In the composition of a quantity of gunpowder, the nitre was 10 lb. more than $\frac{2}{3}$ of the whole, the sulphur was $4\frac{1}{2}$ lb. less than $\frac{1}{6}$ of the whole, and the charcoal was 2 lb. less than $\frac{1}{7}$ of the whole: what was the amount of powder?

12. A lady spent at one store \$1 more than $\frac{1}{2}$ of her money; at another $\$\frac{1}{2}$ more than $\frac{1}{2}$ of what was left; at a third, \$1 more than $\frac{1}{2}$ of what was left, and then had \$1: what did she have at first?

13. Several detachments of artillery divided a certain number of cannon-balls: the first took 72 balls and $\frac{1}{9}$ of the remainder; the second took 144 balls and $\frac{1}{9}$ of the remainder; the third took 216 balls and $\frac{1}{9}$ of the remainder, and so on. The balls were thus divided equally among the detachments: how many balls and how many detachments?

14. Find a number which, when multiplied by 5, 24 taken from the product, the remainder divided by 3, and 28 taken from the quotient, will give the same number.

15. A and B have the same annual income; A saves $\frac{1}{5}$ of his, but B spends annually \$400 more than A; at the end of 4 years B finds himself \$1100 in debt: what is the income of each?

16. What number is as much more than $\frac{3}{4}$ as $\frac{2}{3}$ is less than $\frac{3}{4}$?

17. Five times $\frac{2}{9}$ of what number is 4 less than 8 times $\frac{1}{7}$ of it?

18. Seven times the ratio of what number to $1\frac{5}{8}$ is equal to that number added to $\frac{1}{8}$ of the ratio of 9 to 4?

19. Divide the number 54 into two parts such that $\frac{2}{3}$ of the smaller increased by 12 will equal $\frac{3}{4}$ of the larger diminished by 3.

20. The sum of two numbers is 160; $\frac{3}{4}$ of their difference is 24: what are the numbers?

21. The greater of two numbers divided by their difference equals 21; their sum is 164: what are the numbers?

22. One-half of the ratio of 4 to 6 is equal to $\frac{1}{6}$ of the ratio of 5 to what number?

23. Twelve times the ratio of $\frac{1}{3}$ to $\frac{3}{7}$ is equal to $\frac{1}{3}$ of the ratio of what number to 16?

24. How many times the ratio of $\frac{1}{2}$ to $\frac{3}{4}$ is equal to 3 times the ratio of 1 to 3?

25. A can do a piece of work in 10 days; A and B can do it in 6 days. In what time can B do it alone?

26. A and B can do a piece of work in 16 days; after working 4 days, A leaves B to finish the work, which he does in 36 days. In what time can each do it alone?

B. PROBLEMS OF TWO BASES.

175. Solving Equations of Two Unknown Numbers.—When there are two unknown numbers to be found, there must be two *independent* equations to be used in finding the unknown numbers.

NOTE.—Two equations are independent of each other, when one cannot be obtained from the other alone. Thus,

$$x+y=10 \text{ and}$$
$$x-y=4,$$

are independent; but

$$x+y=10 \text{ and}$$
$$3x+3y=30,$$

are dependent.

Two independent equations, relating the same two unknown

numbers, may be so combined as to form one new equation, containing but one unknown number. This process is called **Elimination.** There are three plans of elimination: (1) by *substitution*, (2) by *comparison*, and (3) by *addition and subtraction*.

I. *By Substitution.*

EXAMPLES.

1. If $x+y=10$ and $y=4$, find the value of x.

Process: (1) $x+y=10$.
(2) $y=4$.

Putting 4 for the y in (1), we have—
(3) $x+4=10$.
Trans., (4) $x=10-4=6$, result.

2. $2x-y=13$ and $\frac{y}{9}=\frac{1}{3}$. Find the values of x and y.

Process: (1) $2x-y=13$.
(2) $\frac{y}{9}=\frac{1}{3}$
$9\times(2)=$(3) $y=\frac{9}{3}=3$, one result.
(1)=(4) $2x-3=13$.
Trans., (5) $2x=13+3=16$.
$\frac{1}{2}$ of (5)=(6) $x=8$, the other result.

3. $x-7y=-38$ and $\frac{2y}{4}=3$. Find the values of x and y.

Process: (1) $x-7y=-38$.
(2) $\frac{2y}{4}=3$.
(3) $y=\frac{3\times4}{2}=6$, one result.
(1)=(4) $x-42=-38$.
Trans., (5) $x=42-38=4$, the other result.

4. $x+y=15$, and $x-y=5$. Find the value of x and y.

Process: (1) $x+y=15$.
(2) $x-y=5$.
Trans., (3) $x=5+y$.
(1)=(4) $(5+y)+y=15$; or,

(5) $5+y+y=15.$

Trans. and col., (6) $2y=15-5=10.$

(7) $y=5.$

(3) = (8) $x=5+5=10.$

5. $3x-4y=10$ and $5x+6y=42$. Find the values of x and y.

Process: (1) $3x-4y=10.$

(2) $5x+6y=42.$

Trans. in (1), (3) $3x=10+4y.$

$\frac{1}{3}$ of (3) = (4) $x=\frac{10+4y}{3}.$

(2) = (5) $5\left(\frac{10+4y}{3}\right)+6y=42.$

$3\times$(5) = (6) $5(10+4y)+18y=126.$

(6) = (7) $50+20y+18y=126.$

Trans., (8) $20y+18y=126-50=76.$

Col., (9) $38y=76.$

(10) $y=2.$

(4) = (11) $x=\frac{10+8}{3}=6.$

6. $\frac{3x+7}{3y-4}=5$ and $\frac{7x-6}{5y+3}=2$. Find the values of x and y.

Process: (1) $\frac{3x+7}{3y-4}=5.$

(2) $\frac{7x-6}{5y+3}=2.$

$(3y-4)\times$(1) = (3) $3x+7=15y-20.$

$(5y+3)\times$(2) = (4) $7x-6=10y+6.$

Trans. in (3), (5) $3x-15y=-27.$

Trans. in (4), (6) $7x=10y+12.$

$\frac{1}{7}$ of (6) = (7) $x=\frac{10y+12}{7}.$

(5) = (8) $3\left(\frac{10y+12}{7}\right)-15y=-27.$

$7\times$(8) = (9) $3(10y+12)-105y=-189.$

(9) = (10) $30y+36-105y=-189.$

Trans. and col., (11) $-75y=-225.$

(12) $y=3.$

(7) = (13) $x=\frac{30+12}{7}=6.$

EXERCISE CLIII.

Find the values of the unknown numbers:

1. $\begin{cases} 2x+y=22. \\ y=6. \end{cases}$

2. $\begin{cases} x-2y=-8. \\ \dfrac{y}{5}=1. \end{cases}$

3. $\begin{cases} 3x-2y=10. \\ \dfrac{5x}{6}=10. \end{cases}$

4. $\begin{cases} \dfrac{25x}{y-x}=15. \\ 3x=9. \end{cases}$

5. $\begin{cases} 2x-3y=-7. \\ x=y. \end{cases}$

6. $\begin{cases} \dfrac{7x+y-4}{8}=5. \\ \dfrac{9x}{5}=y. \end{cases}$

7. $\begin{cases} \dfrac{4x-1}{3x+\frac{1}{2}y}=1. \\ \dfrac{5x-4y}{x-y}=7. \end{cases}$

8. $\begin{cases} x-3+7y=71. \\ \dfrac{5y-12}{x-1}=3. \end{cases}$

II. *By Addition and Subtraction.*

EXAMPLES.

1. $x+y=25$ and $x-y=7$. **Find the values of x and y.**

$$\begin{aligned}
\textit{Process:}\ (1)\ & x+y=25. \\
(2)\ & x-y=7. \\
(1)+(2)=(3)\ & 2x=32. \\
(4)\ & x=16. \\
(1)=(5)\ & 16+y=25. \\
(6)\ & y=25-16=9.
\end{aligned}$$

2. $5x-7y=18$ and $4x+3y=23$. **Find the values of x and y.**

$$\begin{aligned}
\textit{Process:}\ (1)\ & 5x-7y=18. \\
(2)\ & 4x+3y=23. \\
3\times(1)=(3)\ & 15x-21y=54. \\
7\times(2)=(4)\ & 28x+21y=161. \\
(3)+(4)=(5)\ & 43x=215. \\
(6)\ & x=5. \\
(2)=(7)\ & 20+3y=23. \\
\text{Trans.},\ (8)\ & 3y=23-20=3. \\
(9)\ & y=1.
\end{aligned}$$

PLAN: (1) Multiply the equations by such numbers as will make the

coefficient of the x's or the y's equal. (2) If the signs be *unlike, add;* if *alike, subtract.*

Questions: Why did I multiply by 3 and 7 in No. (2)? Could I have made the coefficients of the x's equal by multiplying by 4 and 5? Can the equations be solved in that way? Try it.

3. $5x-3y=37$ and $6x=11y$. Find the values of x and y.

Process: (1) $5x-3y=37$.
(2) $6x=11y$.
Trans., (3) $6x-11y=0$.
$5\times(3)=$(4) $30x-55y=0$.
$6\times(1)=$(5) $30x-18y=222$.
$(4)-(5)=$(6) $-37y=-222$.
(7) $y=6$.
$(2)=$(8) $6x=66$.
(9) $x=11$.

4. $\dfrac{3x+7}{3y-4}=5$ and $\dfrac{7x-6}{5y+3}=2$. Find the values of x and y.

Process: (1) $\dfrac{3x+7}{3y-4}=5$.
(2) $\dfrac{7x-6}{5y+3}=2$.
$(3y-4)\times(1)=$(3) $3x+7=15y-20$.
$(5y+3)\times(2)=$(4) $7x-6=10y+6$.
Trans. in (3), (5) $3x-15y=-27$.
Trans. in (4), (6) $7x-10y=12$.
$2\times(5)=$(7) $6x-30y=-54$.
$3\times(6)=$(8) $21x-30y=36$.
$(7)-(8)=$(9) $-15x=-90$.
(10) $x=6$.
$(5)=$(11) $18-15y=-27$.
(12) $-15y=-18-27=-45$.
(13) $y=3$.

EXERCISE CLIV.

Find the values of x *and* y:

1. $\begin{cases} x+y=29. \\ x-y=17. \end{cases}$

2. $\begin{cases} 3x+y=22. \\ x+y=12. \end{cases}$

3. $\begin{cases} 3x-2y=-1. \\ 4x=3y. \end{cases}$

4. $\begin{cases} 5x-7y=-10. \\ x=y. \end{cases}$

5. $\begin{cases} \dfrac{x}{4}-\dfrac{y}{8}=0. \\ \dfrac{x}{8}+\dfrac{y}{3}=7. \end{cases}$

6. $\begin{cases} 5x-\dfrac{3y}{9}=60. \\ x+y=28. \end{cases}$

7. $\begin{cases} 16x-7\frac{1}{2}y=50. \\ \dfrac{3x+5}{2y-3}=4. \end{cases}$

8. $\begin{cases} \dfrac{x+5y}{13}-\dfrac{2y+x}{11}=-1. \\ 3x-y=2. \end{cases}$

III. *By Comparison.*

EXAMPLES.

1. $x+y=12$ and $x-y=6$. Find the values of x and y.

Process : (1) $x+y=12$.
(2) $x-y=6$.
Transpose in (1), (3) $x=12-y$.
Transpose in (2), (4) $x=6+y$.
(5) $6+y=12-y$. (Why ?)
Transpose, (6) $y+y=12-6=6$.
(7) $2y=6$.
(8) $y=3$.
(4) = (9) $x=6+3=9$.

2. $3x+2y=17$ and $4x+y=16$. Find the values of x and y.

Process : (1) $3x+2y=17$.
(2) $4x+y=16$.
Trans. in (1), (3) $2y=17-3x$.
$\frac{1}{2}$ of (3) = (4) $y=\dfrac{17-3x}{2}$.
Trans. in (2), (5) $y=16-4x$.
(6) $16-4x=\dfrac{17-3x}{2}$. (Why ?)
$2\times(6)$ = (7) $32-8x=17-3x$.
Trans., (8) $3x-8x=17-32$.
Col., (9) $-5x=-15$.
(10) $x=3$.
(5) = (11) $y=16-12=4$.

PLAN: Solve each equation for y in terms of x; then, make those

values of y equal. This gives equation (6), containing one unknown number, x.

Questions: Could I have solved for values of x in terms of y? Would this be correct? Try it.

3. $\frac{x}{3}+\frac{z}{4}=8$ and $x-z=-3$. Find the values of x and z.

Process: (1) $\frac{x}{3}+\frac{z}{4}=8$.

(2) $x-z=-3$.

Trans. in (1), (3) $\frac{x}{3}=8-\frac{z}{4}$.

$3\times(3)=$ (4) $x=24-\frac{3z}{4}$.

Trans. in (2), (5) $x=z-3$.

(6) $z-3=24-\frac{3z}{4}$. (Why?)

$4\times(6)=$ (7) $4z-12=96-3z$.

Trans., (8) $4z+3z=96+12$.

Col., (9) $7z=108$.

(10) $z=15\frac{3}{7}$.

(5) = (11) $x=15\frac{3}{7}-3=12\frac{3}{7}$.

4. $5x-3y=34$ and $5y-3x=2$. Find the values of x and y.

Process: (1) $5x-3y=34$.

(2) $5y-3x=2$.

Trans. in (1), (3) $5x=34+3y$.

$\frac{1}{5}$ of (3) = (4) $x=\frac{34+3y}{5}$.

Trans. in (2), (5) $-3x=2-5y$.

Changing signs, (6) $3x=5y-2$.

$\frac{1}{3}$ of (6) = (7) $x=\frac{5y-2}{3}$.

(8) $\frac{5y-2}{3}=\frac{34+3y}{5}$. (Why?)

$15\times(8)=$ (9) $25y-10=102+9y$.

Trans., (10) $25y-9y=102+10$.

Col., (11) $16y=112$.

(12) $y=7$.

(7) = (13) $x=\frac{35-2}{3}=11$.

5. $\frac{3}{x}+\frac{3}{y}=4.\quad \frac{9}{x}-\frac{1}{y}=8\tfrac{2}{3}.$ Find the values of x and y.

$$
\begin{array}{rl}
\textit{Process}:\ (1) & \frac{3}{x}+\frac{3}{y}=4. \\
(2) & \frac{9}{x}-\frac{1}{y}=8\tfrac{2}{3}. \\
\text{Trans. in (1), } (3) & \frac{3}{x}=4-\frac{3}{y}. \\
\tfrac{1}{3}\text{ of (3)}=(4) & \frac{1}{x}=\frac{4}{3}-\frac{1}{y}. \\
\text{Trans. in (2), } (5) & \frac{9}{x}=\frac{26}{3}+\frac{1}{y}. \\
\tfrac{1}{9}\text{ of (5)}=(6) & \frac{1}{x}=\frac{26}{27}+\frac{1}{9y}. \\
(7) & \frac{4}{3}-\frac{1}{y}=\frac{26}{27}+\frac{1}{9y}. \\
27y\times(7)=(8) & 36y-27=26y+3. \\
\text{Trans. and col., } (9) & 10y=30. \\
(10) & y=3. \\
(4)=(11) & \frac{1}{x}=\frac{4}{3}-\frac{1}{3}=1. \\
(12) & x=1.
\end{array}
$$

NOTE.—Instead of expressing the values of x in terms of y, I expressed values of $\frac{1}{x}$ in terms of y. This is the more convenient plan.

EXERCISE CLV.

Find the values of x and y.

1. $\begin{cases} 7x+3y=59. \\ x+5y=45. \end{cases}$

2. $\begin{cases} 7x-5y=12. \\ 2x+y=18. \end{cases}$

3. $\begin{cases} 4x+2y=15. \\ 3x-4y=-8. \end{cases}$

4. $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{7}{12}. \\ \frac{12}{x}+\frac{18}{y}=9. \end{cases}$

5. $\begin{cases} \frac{3}{x}+\frac{5}{y}=2. \\ \frac{1}{x}+\frac{1}{y}=\frac{8}{15}. \end{cases}$

6. $\begin{cases} \frac{x-y}{3}=\frac{3x-6}{10}. \\ x-2y=6. \end{cases}$

7. $\begin{cases} \frac{11x}{5}=\frac{6y+22}{4}. \\ 3x-4y=29. \end{cases}$

176. Problems.

EXAMPLES.

1. The sum of two numbers is 50, their difference is 30: find the numbers.

Solution: Let x = one number and
y = the other number.

Then, (1) $x+y=50$
and (2) $x-y=30$.
(1)+(2)=(3) $2x=80$.
(4) $x=40$.
(1)=(5) $40+y=50$.
(6) $y=50-40=10$.
Answer, 40 and 10.

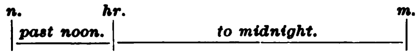
2. A and B together have $1500; A has twice as much as B. How much has each?

Solution: Let x = A's money and
y = B's money.

Then, (1) $x+y=1500$
and (2) $x=2y$.
(1)=(3) $2y+y=1500$.
(4) $3y=1500$.
(5) $y=500$.
(2)=(6) $x=1000$.
Answer: A, $1000; and B, $500.

3. If $\frac{2}{3}$ of the time past noon is equal to $\frac{2}{9}$ of the time to midnight, what is the hour?

NOTE.—The time past noon plus the time to midnight is 12 hours.



Solution: Let x = time past noon and
y = time to midnight.

Then, (1) $x+y=12$
and (2) $\frac{2}{3}x=\frac{2}{9}y$.
$\frac{3}{2}\times$(2)=(3) $x=\frac{1}{3}y$.
(1)=(4) $\frac{1}{3}y+y=12$.
(5) $\frac{4}{3}y=12$.
(6) $y=9$.
(3)=(7) $x=3$.

As the time past noon is the hour, the answer is 3 p. m.

4. Divide 12 into two such parts that 3 times the one and 5 times the other shall be 46.

Solution: Let x = the larger part and
y = the smaller part.

Then, (1) $x+y=12$
and (2) $3x+5y=46$.
$3\times(1)=(3)$ $3x+3y=36$.
$(2)-(3)=(4)$ $2y=10$.
(5) $y=5$.
$(1)=(6)$ $x+5=12$.
(7) $x=12-5=7$.
Answer, 7 and 5.

5. A and B had \$30 each; after B paid A a certain debt, A then had twice as much as B. How much has each?

Solution: Let x = A's money at close and
y = B's money at close.

Then, (1) $x+y=60$
and (2) $x=2y$.
$(1)=(3)$ $2y+y=60$.
(4) $3y=60$.
(5) $y=20$.
$(1)=(6)$ $x+20=60$.
(7) $x=40$.
Answer, \$40 and \$20.

6. A certain number is composed of two digits; the sum of the units and tens is 6; and if 3 times the digit in tens' place be subtracted from 4 times that in units' place, the remainder will be minus 4. Find the number.

Solution: Let x = the units' figure and
y = the tens' figure.

Then (1) $x+y=6$
and (2) $4x-3y=-4$.
$3\times(1)=(3)$ $3x+3y=18$.
$(2)+(3)=(4)$ $7x=14$.
(5) $x=2$.
(6) $y=6-2=4$.
Number, 42.

7. Thomas Reed bought 6% mining stock at $114\frac{1}{2}$%, and 4 furnace stock at 112%, brokerage $\frac{1}{2}$% in each case; the latt cost him $430 more than the former, but yields the same annu income. What did each cost?

NOTE.—The cost is the market value plus the brokerage. Then, t cost of the furnace stock is $112\frac{1}{2}$%, or $\frac{9}{8}$ of furnace stock; and the cost the mining stock is 115%, or $\frac{23}{20}$ of mining stock.

Solution: Let x = par value of furnace stock
and y = par value of the mining stock.

Then (1) $\frac{9}{8}x - \frac{23}{20}y = 430$ (why?)

and (2) $\frac{6y}{100} = \frac{4x}{100}$. (Why?)

$100 \times (2) = (3)$ $6y = 4x$.

$40 \times (1) = (4)$ $45x - 46y = 17200$.

$2 \times (4) = (5)$ $90x - 92y = 34400$.

Trans. in (3), (6) $-4x + 6y = 0$.

$22\frac{1}{2} \times (6) = (7)$ $-90x + 135y = 0$.

$(5) + (7) = (8)$ $43y = 34400$.

(9) $y = 800$.

(10) $\frac{23}{20}y = 920$.

$(3) = (11)$ $4x = 4800$.

(12) $x = 1200$.

(13) $\frac{9x}{8} = 1350$.

Answer, $1350 and $920.

8. 5 apples and 4 oranges cost 22¢, 3 apples and 7 orang cost 27¢. Find the cost of 1 apple and 1 orange.

Solution: Let x = cost of 1 apple and
y = cost of 1 orange.

Then, (1) $5x + 4y = 22$ and

(2) $3x + 7y = 27$.

$3 \times (1) = (3)$ $15x + 12y = 66$.

$5 \times (2) = (4)$ $15x + 35y = 135$.

$(4) - (3) = (5)$ $23y = 69$.

(6) $y = 3$.

$(2) = (7)$ $3x + 21 = 27$.

(8) $3x = 6$.

(9) $x = 2$.

Answer, 2¢ and 3¢.

9. A and B can do a piece of work in 16 days; after working 4 days, A leaves B to finish the work, which he does in 36 days. In what time can each do it alone?

Solution: Let x = time reqd. by A to do the work
and y = time reqd. by B to do the work.

Then, $\frac{1}{x}$ = part A can do in 1 da.,

$\frac{1}{y}$ = part B can do in 1 da.,

(1) $\frac{16}{x}+\frac{16}{y}=1$, (why?)

and (2) $\frac{4}{x}+\frac{4}{y}+\frac{36}{y}=1$. (Why?)

$4\times(2)=$ (3) $\frac{16}{x}+\frac{160}{y}=4$.

$(3)-(1)=$ (4) $\frac{144}{y}=3$.

(5) $3y=144$.

(6) $y=48$.

$(2)=$ (7) $\frac{4}{x}+\frac{40}{48}=1$.

(8) $\frac{4}{x}=1-\frac{40}{48}=\frac{1}{6}$.

(9) $4=\frac{x}{3}$.

(10) $x=24$.

Answer, 24 da. and 48 da.

10. Divide \$600 into two parts, having the ratio of $\frac{1}{4}$ to $\frac{1}{6}$.

Solution: Let x = larger part and
y = smaller part.

Then, (1) $x+y=600$

and (2) $x:y::\frac{1}{4}:\frac{1}{6}$.

$(2)=$ (3) $\frac{x}{y}=\frac{6}{4}$.

$y\times(3)=$ (4) $x=\frac{6y}{4}$.

Trans. in (1), (5) $x=600-y$.

(6) $\frac{6y}{4}=600-y$. (Why?)

(7) $6y = 2400 - 4y$.
(8) $6y + 4y = 2400$.
(9) $10y = 2400$.
(10) $y = 240$.
(4) = (11) $x = \frac{6 \times 240}{4} = 360$.
Answer, \$360 and \$240.

11. At what time between 4 and 5 o'clock are the hour-hand and minute-hand of a clock together?

NOTE.—At 4 o'clock the minute-hand is at XII and the hour-hand is at IIII. Then, when the minute-hand gains 20 minute spaces on the hour-hand they will be together.

Solution: Let x = the minute-hand's travel
and y = the hour-hand's travel.
Then, (1) $x - y = 20$
and (2) $x = 12y$. (Why?)
Trans. in (1), (3) $x = 20 + y$.
(4) $12y = 20 + y$. (Why?)
(5) $12y - y = 20$.
(6) $11y = 20$.
(7) $y = \frac{20}{11}$.
(8) $x = \frac{12 \times 20}{11} = 21\frac{9}{11}$.
Answer, $21\frac{9}{11}$ min. after 4 o'clock.

12. At what time between 4 and 5 o'clock are the hour-hand and the minute-hand equally distant from V?

NOTE.—There are two answers to this problem: (1) When the two hands are together, the conditions are the same as in Example 4; (2) when the hands are on opposite sides of V. We will solve for the second answer.

Draw a clock-face and verify this statement: When the hands reach the required places, the minute-hand will lack as far of being to VI as the hour-hand is past IIII.

Solution: Let x = min.-hand's travel
and y = hour-hand's travel.
Then, (1) $x = 30 - y$ (statement above)
and (2) $x = 12y$. (Why?)
(3) $12y = 30 - y$. (Why?)
(4) $13y = 30$.
(5) $y = \frac{30}{13}$.
(2) = (6) $x = \frac{12 \times 30}{13} = 27\frac{9}{13}$.
Answer, $27\frac{9}{13}$ min. after 4 o'clock.

13. The head of a fish weighs 9 pounds, the tail weighs as much as the head and half the body, and the body weighs as much as the head and tail together. Find the weight of the fish.

Solution: Let x = weight of body
and y = weight of tail.
Then, (1) $y=9+\frac{1}{2}x$
and (2) $x=y+9$.
Trans. in (2) (3) $y=x-9$.
(4) $x-9=9+\frac{1}{2}x$. (Why?)
Trans., (5) $x-\frac{1}{2}x=9+9=18$.
(6) $x=36$.
(7) $y=36-9=27$.
Answer = 36 lb. + 27 lb. + 9 lb. = 72 lb.

14. A and B rent a field for $27; A puts in 4 horses for 5 months, and B puts in 10 cows for 6 months: what ought each to pay if 2 horses eat as much as 3 cows?

Solution: (1) Pasture for 2 h. = pasture for 3 c.
$2\times(1)=(2)$ Pasture for 4 h. = pasture for 6 c.
$\therefore$ A's 4 horses are equivalent to 6 cows.
(3) Pasture of 6 c. for 5 mo. = pasture of 30 c. for 1 mo.
(4) Pasture of 10 c. for 6 mo. = pasture of 60 c. for 1 mo.
$\therefore$ they should pay in the ratio of 30 to 60.
Let x = A's part and
y = B's part.
Then, (5) $x+y=27$
and (6) $x:y::30:60$.
$(6)=(7)$ $\frac{x}{y}=\frac{30}{60}=\frac{1}{2}$.
(8) $2x=y$.
$(5)=(9)$ $x+2x=27$.
(10) $x=9$.
(11) $y=18$.
Answer, $9 and $18.

15. A fox is 70 leaps in advance of a hound; the fox takes 8 leaps to the hound's 6, but 2 of the hound's leaps equal 5 of

the fox's: how many leaps must the hound take to catch the fox?

Solution: (1) 5 f. l. = 2 h. l.
$\frac{1}{5}$ of (1) = (2) 1 f. l. = $\frac{2}{5}$ h. l.
$8\times(2)$ = (3) 8 f. l. = $3\frac{1}{5}$ h. l.
$70\times(2)$ = (4) 70 f. l. = 28 h. l.

NOTE.—Re-writing the problem, putting for fox-leaps their equivalents found in (3) and (4), the leaps will all be of the same length. We have, "A fox is 28 leaps ahead of a hound, and takes $3\frac{1}{5}$ leaps while the hound takes 6. How many leaps will the hound make in overtaking the fox, the leaps being all of the same length?"

Let x = No. of leaps for the fox
and y = No. of leaps for the hound.
Then, (5) $y - x = 28$
and (6) $x : y :: 3\frac{1}{5} : 6$.
(6) = (7) $\frac{x}{y} = \frac{16}{30} = \frac{8}{15}$.
(8) $x = \frac{8y}{15}$.
Trans. in (5), (9) $x = y - 28$.
(10) $y - 28 = \frac{8y}{15}$.
$15\times(10)$ = (11) $15y - 420 = 8y$.
Trans., (12) $15y - 8y = 420$.
(13) $y = 60$.
Answer, 60 leaps.

NOTE.—If fox-leaps had been required in the answer, the *hound-leaps* should have been reduced to *fox-leaps*.

16. A banker owns $2\frac{1}{2}\%$ stocks bought at 10% below par, and 3% stocks bought at 15% below par. The income from the former is $66\frac{2}{3}\%$ more than from the latter, and the investment in the latter is \$11400 less than in the former. Find the investment in each.

Solution: Let x = par value of 1st stocks
and y = par value of 2d stocks.
Then, (1) $\frac{2\frac{1}{2}x}{100} = 1\frac{2}{3} \times \frac{3y}{100} = \frac{5y}{100}$ (Why?)
and (2) $\frac{90x}{100} - \frac{85y}{100} = 11400$. (Why?)
$100\times(1)$ = (3) $2\frac{1}{2}x = 5y$.
$100\times(2)$ = (4) $90x - 85y = 1140000$.
(3) $\div 2\frac{1}{2}$ = (5) $x = 2y$.
(6) $180y - 85y = 1140000$. (Why?)
(7) $95y = 1140000$.
(8) $y = 12000$.
(9) $\frac{85y}{100} = 10200$.
(5) = (10) $x = 24000$.
(11) $\frac{90x}{100} = 21600$.
Answer, \$10200 and \$21600.

17. My agent sold my flour at 4% commission; increasing the proceeds by \$4.20, I ordered the purchase of wheat at 2% commission; after which, wheat declining $3\frac{1}{3}\%$, my whole loss was \$5. What was the selling price of the flour?

Solution: Let x = selling price of flour
and y = cost price of wheat.

Then, $\frac{96\,x}{100}$ = proceeds from flour,

$\frac{96\,x}{100}+4.2$ = investment in wheat,

$$(1)\quad \frac{102\,y}{100}=\frac{96\,x}{100}+4.2, \text{ (why ?)}$$

$$\text{and } (2)\quad \frac{96\frac{2}{3}\,y}{100}=x+4.2-5. \text{ (Why ?)}$$

$$100\times(1)=(3)\quad 102\,y=96\,x+420.$$
$$300\times(2)=(4)\quad 290\,y=300\,x+1260-1500.$$
$$(5)\quad 29\,y=30\,x-24.$$
$$\tfrac{1}{3}\text{ of }(3)=(6)\quad 34\,y=32\,x+140.$$
$$16\times(5)=(7)\quad 464\,y=480\,x-384.$$
$$15\times(6)=(8)\quad 510\,y=480\,x+2100.$$
$$(8)-(7)=(9)\quad 46\,y=2484.$$
$$(10)\quad y=54.$$
$$\text{Trans. in (2), } (11)\quad x=\frac{96\frac{2}{3}\,y}{100}+5-4.2.$$
$$(12)\quad x=\frac{96\frac{2}{3}\times 54}{100}+5-4.2=53. \quad \text{Answer, \$53.}$$

18. Suppose 10% state stock 20% better in market than 4% railroad stock. If A's income be \$500 from each, how much has he paid for each, the whole investment bringing $6\frac{2}{333}\%$?

NOTE.—The solution to this problem is given in four steps: (1) We obtain the par value of the state stock; (2) the par value of the railroad stock; (3) the amount of the investment; and (4) with the results obtained from the first three steps we are enabled to obtain the bases for the last step, which, when solved, gives required results.

Step 1: (1) 10% s. s. = \$500, income on s. s. (Basis.)
$10\times(1)=(2)$ 100% s. s. = \$5000, par value of s. s.
Step 2: (1) 4% r. r. s. = \$500, income on r. r. s. (Basis.)
$25\times(1)=(2)$ 100% r. r. s. = \$12500, par value of r. r. s.

Step 3: (1) $6\frac{2}{333}\%$ inv. = \$1000, whole income. (Basis.)
$\frac{333}{2000}$ of (1) = (2) 1% inv. = \$166.50.
$100 \times (2) = (3)$ 100% inv. = \$16650.

NOTE.—There are 50 shares state stock and 125 shares r. r. stock.

Step 4: Let x = market value of 1 share s. s.
and y = market value of 1 share r. r. s.
Then, (1) $50x = \frac{120}{100} \times 50y$ (why?)
and (2) $50x + 125y = 16650$.
Trans. in (1), (3) $50x - 60y = 0$.
$(2) - (3) = (4)$ $185y = 16650$.
(5) $y = 90$.
(6) $5x = 540$.
(7) $x = 108$.

State stock: 50 sh. at 108% = \$5400, answer.
R. R. stock: 125 sh. at 90% = \$11250, answer.

EXERCISE CLVI.

1. Divide \$500 between A and B, so that A may have \$40 more than B.

2. Divide \$420 between Charles and Henry, so that Henry will have \$25 less than Charles.

3. Two men owe a debt of \$54 in the ratio of 4 to 5: what does each owe?

4. Two men owe a debt; for every dollar that A owes, B owes \$1.50. If the debt is \$250, find each man's part.

5. Divide 20 into two such parts that the smaller plus 5 shall equal the larger minus 5.

6. A and B have \$500; A says to B, "Give me \$50 and then I will have as much as you will have." How much has each?

7. James and John bought a watermelon for \$.35; they divided it so that John got $2\frac{1}{2}$ times as much as James: what should each pay?

8. $\frac{1}{2}$ of the time past noon is equal to $\frac{1}{5}$ of the time to midnight: what is the time?

9. $\frac{1}{4}$ of the time past noon plus $3\frac{1}{2}$ hours is equal to $\frac{2}{3}$ of the time to midnight. Find the hour.

10. $\frac{1}{3}$ of the time past midnight plus 2 hours is equal to $\frac{1}{2}$ of the time to noon minus $1\frac{1}{2}$ hours: find the time past midnight.

11. All the time past midnight minus 2 hours is equal to $\frac{1}{4}$ of the time to noon. What is the time?

12. A number is composed of 2 digits; the sum of the digits is 5, and the number of units is $\frac{2}{3}$ of the number of tens. Find the number.

13. A number is composed of 2 digits; the sum of the digits is 12; 5 times the number of units is equal to $2\frac{1}{2}$ times the number of tens. Find the number.

14. A number has two places; the sum of the units and tens is 4; if 36 be subtracted, the order will be reversed. Find the number.

NOTE.—Any number expressed by the French system of notation is equal to the sum of its 1st (units') digit, 10 times its 2d (tens') digit, 100 times its 3d (hundreds') digit, and so on. Convince yourself of this truth by applying it to several numbers.

15. A and B are partners; A invests a capital of $500, and B a capital of $850; the gain is $540: divide it.

16. A and B are partners; they invested capital in the ratio of 5 to 8; they have gained $650: divide it.

17. Divide $\frac{1}{4}$ in the ratio of 6 to 7.

18. Divide .06 in the ratio of $\frac{1}{2}$ to $\frac{1}{4}$.

19. Divide 10.6 in the ratio of .1 and .05.

20. A and B are partners; A puts in $660 for 10 months, and B has in $500 for the first 6 months and only $100 for the other 4 months; they gain $600: divide it.

21. A and B have a joint stock of $2000 by which they gain $640, of which A receives $128 more than B: what is each man's share of the stock?

22. How many minutes does it lack of 4 o'clock, if $\frac{3}{4}$ of an hour ago it was twice as many minutes past 2 o'clock?

23. A and B start at the same time from two places, M and N, 154 miles apart, and each travels toward the other till they meet; A travels 3 miles in 2 hours, and B travels 5 miles in 4 hours: where will they meet?

NOTE.—In 1 hour A travels $1\frac{1}{2}$ miles and B travels $1\frac{1}{4}$ miles; therefore the distance is to be divided in the ratio of $1\frac{1}{2}$ to $1\frac{1}{4}$.

24. A house and garden cost $850; 5 times the price of the house equals 12 times the cost of the garden: find the cost of each.

25. The sum of two numbers is 5760, and their difference is equal to $\frac{1}{8}$ of the greater: find the numbers.

26. A agrees to work for $2 a day and forfeit $1 for every day he is idle; at the end of 20 days he receives $25: how many days does he idle?

NOTE.—To receive $25 he must work $12\frac{1}{2}$ days. During the remaining $7\frac{1}{2}$ days he forfeited all he earned; therefore he idled twice as much as he worked. The problem then becomes: Divide $7\frac{1}{2}$ into two parts in the ratio of 1 to 2.

27. A has 7 loaves of bread, B 5, and C none. The three eat all of the bread, each the same amount; C pays to A and B 12 cents: what should each receive?

NOTE.—Each will eat one-third of the twelve loaves, or 4 loaves. A eats 4, and has 3 left for C; B eats 4, and has 1 left for C. C therefore eats 3 of A's and 1 of B's, and the 12 cents should be divided in the ratio of 3 to 1.

28. Two boys run a race: the smaller boy steps 4 feet and the larger steps 6 feet; and the smaller boy takes 5 steps while the larger boy takes 4; the larger boy gives the smaller 120 feet the start, and they come out even: how many steps does the larger boy take, and how far does he run?

29. In the last number, how many steps does the smaller boy take, and how far does he run?

30. Two cog-wheels work together; one has 11 cogs and the other 35 cogs: in how many rounds of the large wheel will the smaller gain 72 rounds?

31. A hare is 30 leaps in advance of a greyhound; the hare makes 11 leaps while the hound makes 9, and leaps four-fifths as far as the hound: how many more leaps will the hare make before the hound catches him?

32. There are two numbers, the one twice as large as the other; one-third of the smaller and one-half of the larger equal 20: find the numbers.

33. Four-fifths of A's money is equal to one-half of B's; A gains $200 and B loses $100; they now have the same: what had they at first?

34. A number is composed of 2 digits; the number is equal to 9 times the units, less 18; it is also equal to 12 times the difference between the units and the tens: find the number.

35. A number is composed of two digits: the difference between the units and the tens is 4; 5 times the units is equal to $2\frac{1}{2}$ times the tens: find the number.

36. A is 4 times as old as his son; in 14 years he will be only twice as old as the son: find their ages.

37. At what time between 8 o'clock and 9 o'clock are the hands of a clock together?

38. At what time between 8 o'clock and 9 o'clock are the hands of a clock opposite each other?

39. At what times between 5 and 6 is the minute-hand half-way between the hour-hand and XII?

40. At what times between 5 and 6 will the hands be at right angles to each other?

41. If one-half of the time past midnight is equal to one-sixth of the time past noon, what is the hour?

42. If one-fourth of the time past noon is equal to one-sixteenth of the time past midnight, what is the hour?

43. If the time to noon equals one-seventh of the time to midnight, what is the hour?

44. If one-half of the time to midnight is equal to one-eighth of the time to noon, what is the hour?

45. A farmer sold to one man 20 bushels of wheat and 8 bushels of oats for $14; to another 12 bushels of wheat and 10 bushels of oats for $9.70: find the price of the wheat and the oats.

46. A man bought 3 cows and 7 calves for $95; again, he bought 5 cows and 4 calves for $120: find the price of a cow and a calf.

47. A is 30 years old; A's age is equal to B's plus one-half of C's; and C is as old as A and B: find the ages of B and C.

48. The head of a fish is 6 inches long; its tail is as long as the head and one-fourth of the body; and the body is as long as the head and the tail: find the length of the fish.

49. A bag contains three times as many dollars as quarters; if 8 dollars and 8 quarters be taken away, there will be 5 times as many dollars as quarters: find the number of each.

50. A has two horses and a saddle; the saddle is worth $20; if it be placed on one horse it will make him worth as much as the other; but if it be placed on the second horse, he is then worth twice as much as the first: find the value of each horse.

51. A flag-pole consists of two parts; the length of the upper is five-sevenths of the length of the lower; and 9 times the upper part added to 13 times the lower part is longer than 11 times the whole pole by 36 feet; find the length of the pole.

52. A party was composed of men and women; 6 of the women left; there were then twice as many men as women; when the 6 women returned with their husbands, the number of women was only two-thirds of the number of men. What was the size of the original party?

30. Two cog-wheels work together; one has 11 cogs and the other 35 cogs: in how many rounds of the large wheel will the smaller gain 72 rounds?

31. A hare is 30 leaps in advance of a greyhound; the hare makes 11 leaps while the hound makes 9, and leaps four-fifths as far as the hound: how many more leaps will the hare make before the hound catches him?

32. There are two numbers, the one twice as large as the other; one-third of the smaller and one-half of the larger equal 20: find the numbers.

33. Four-fifths of A's money is equal to one-half of B's; A gains $200 and B loses $100; they now have the same: what had they at first?

34. A number is composed of 2 digits; the number is equal to 9 times the units, less 18; it is also equal to 12 times the difference between the units and the tens: find the number.

35. A number is composed of two digits: the difference between the units and the tens is 4; 5 times the units is equal to $2\frac{1}{2}$ times the tens: find the number.

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41. If one-half of the time past midnight is equal to one-sixth of the time past noon, what is the hour?

42. If one-fourth of the time past noon is equal to one-sixteenth of the time past midnight, what is the hour?

43. If the time to noon equals one-seventh of the time to midnight, what is the hour?

44. If one-half of the time to midnight is equal to one-eighth of the time to noon, what is the hour?

45. A farmer sold to one man 20 bushels of wheat and 8 bushels of oats for \$14; to another 12 bushels of wheat and 10 bushels of oats for \$9.70: find the price of the wheat and the oats.

46. A man bought 3 cows and 7 calves for \$95; again, he bought 5 cows and 4 calves for \$120: find the price of a cow and a calf.

47. A is 30 years old; A's age is equal to B's plus one-half of C's; and C is as old as A and B: find the ages of B and C.

48. The head of a fish is 6 inches long; its tail is as long as the head and one-fourth of the body; and the body is as long as the head and the tail: find the length of the fish.

49. A bag contains three times as many dollars as quarters; if 8 dollars and 8 quarters be taken away, there will be 5 times as many dollars as quarters: find the number of each.

50. A has two horses and a saddle; the saddle is worth \$20; if it be placed on one horse it will make him worth as much as the other; but if it be placed on the second horse, he is then worth twice as much as the first: find the value of each horse.

51. A flag-pole consists of two parts; the length of the upper is five-sevenths of the length of the lower; and 9 times the upper part added to 13 times the lower part is longer than 11 times the whole pole by 36 feet; find the length of the pole.

52. A party was composed of men and women; 6 of the women left; there were then twice as many men as women; when the 6 women returned with their husbands, the number of women was only two-thirds of the number of men. What was the size of the original party?

53. Twenty-six tons can be carried by 15 wagons and 22 carts, or by 18 wagons and 16 carts: what is a load for a cart and for a wagon?

54. A and B started from M and N respectively, and traveled till they met; it appeared that A had traveled five-sevenths as far as B; but if A had traveled 15 miles farther, he would have traveled twice as far as B: how far did each travel?

55. A cistern is filled by one pipe and emptied by another. Their capacities for carrying water are to each other as 6:5. If both pipes are left open, the cistern will fill in 90 min. Find the time required by the first alone to fill the cistern, and the time required by the second alone to empty it.

56. A cistern is filled by two pipes in 75 min. Their capacities for carrying water are to each other as 5:3. In what time can each alone fill the cistern?

57. A, after doing $\frac{1}{4}$ of a work in 5 days, calls the assistance of B, and they finish the work in 6 days: in what time could each have done the whole work?

58. A and B could have done a work in 15 days, but after working together 6 days, B was left to finish it, which he did in 30 days: in what time could A have finished it if B had left at the end of the 6 days?

C. PROBLEMS OF THREE OR MORE BASES.

177. Solving Equations of Three or More Unknown Numbers.—When there are three or more unknown numbers to be found, there must be as many equations (bases) as there are unknown numbers.

The process of elimination here is not different from that given under *Problems of Two Bases.*

EXAMPLES.

1. $3x+5y-z=18$, $5x-6y+5z=8$, $x+y+z=6$. Find the values of x, y, and z.

Process:
$$
\begin{aligned}
&(1)\ 3x+5y-z=18.\\
&(2)\ 5x-6y+5z=8.\\
&(3)\ x+y+z=6.\\
3\times(3)=&(4)\ 3x+3y+3z=18.\\
(1)-(4)=&(5)\ 2y-4z=0.\\
5\times(3)=&(6)\ 5x+5y+5z=30.\\
(6)-(2)=&(7)\ 11y=22.\\
&(8)\ y=2.\\
(5)=&(9)\ 4-4z=0.\\
&(10)\ z=1.\\
(3)=&(11)\ x+2+1=6.\\
&(12)\ x=3.
\end{aligned}
$$

2. $x+y+z=22$, $y+z+r=21$, $x+z+r=19$, $x+y+r=16$. Find the values of x, y, z, and r.

Process:
$$
\begin{aligned}
&(1)\ x+y+z=22.\\
&(2)\ y+z+r=21.\\
&(3)\ x+z+r=19.\\
&(4)\ \underline{x+y+r=16.}\\
\text{Adding, }&(5)\ 3x+3y+3z+3r=78.\\
\tfrac{1}{3}\text{ of }(5)=&(6)\ x+y+z+r=26.\\
(6)-(1)=&(7)\ r=4.\\
(6)-(2)=&(8)\ x=5.\\
(6)-(3)=&(9)\ y=7.\\
(6)-(4)=&(10)\ z=10.
\end{aligned}
$$

3. $3x+2y=13$, $3y-2z=8$, $2x-3z=9$. Find the values of x, y, and z.

Process:
$$
\begin{aligned}
&(1)\ 3x+2y=13.\\
&(2)\ 3y-2z=8.\\
&(3)\ 2x-3z=9.\\
3\times(1)=&(4)\ 9x+6y=39.\\
2\times(2)=&(5)\ 6y-4z=16.\\
(4)-(5)=&(6)\ 9x+4z+23.\\
2\times(6)=&(7)\ 18x+8z=46.\\
9\times(3)=&(8)\ 18x-27z=81.\\
(8)-(7)=&(9)\ -35z=35.\\
&(10)\ z=-1.\\
(2)=&(11)\ 3y+2=8.\\
&(12)\ y=2.\\
(1)=&(13)\ 3x+4=13.\\
&(14)\ x=3.
\end{aligned}
$$

EXERCISE CLVII.

Find the values of the unknown numbers:

1. $\begin{cases} 2x+y=13. \\ x+3z=11. \\ y+z=5. \end{cases}$

2. $\begin{cases} x+y+z=11. \\ 2x-y+4z=26. \\ 6x-z=-14. \end{cases}$

3. $\begin{cases} x+3y+2z=17. \\ 4x-2y+z=12. \\ 3x-y=10. \end{cases}$

4. $\begin{cases} 6x+5y+4z=37. \\ 7x-3y+3z=26. \\ 3x+2y-z=3. \end{cases}$

5. $\begin{cases} x+y+z=7. \\ y+z+w=6. \\ z+w+x=8. \\ w+x+y=9. \end{cases}$

6. $\begin{cases} 3x+y=11. \\ 4y+6z=2. \\ 7z+5r=11. \\ 3r+2x=4. \end{cases}$

7. $\begin{cases} 8x-10y+4z=60. \\ 7x+3y-5z=23. \\ x-2y+4z=36. \end{cases}$

8. $\begin{cases} \frac{x}{6}+\frac{y}{8}+\frac{z}{3}=9. \\ \frac{5x}{4}+\frac{3y}{4}-\frac{4z}{5}=39. \\ x+y+z=43. \end{cases}$

178. Problems.

EXAMPLES.

1. A, B and C have \$210: twice A's money and 3 times B's money added to C's will make \$400; A's money and 4 times B's added to twice C's will make \$510. How much has each?

Solution: Let x = A's money,
y = B's money, and
z = C's money.

$$\begin{array}{rl}
\text{Then, (1)} & x+y+z=210, \\
(2) & 2x+3y+z=400, \\
\text{and (3)} & x+4y+2z=510. \\
(3)-(1)=(4) & 3y+z=300. \\
2\times(1)=(5) & 2x+2y+2z=420. \\
(5)-(2)=(6) & -y+z=20. \\
(4)-(6)=(7) & 4y=280. \\
(8) & y=70. \\
(6)=(9) & -70+z=20. \\
(10) & z=90. \\
(1)=(11) & x+70+90=210. \\
(12) & x=50.
\end{array}$$

Answer, \$50, \$70, and \$90.

2. A and B together have \$600; A and C, \$700; B and C, \$500. How much has each?

Solution: Let $x =$ A's money,
$y =$ B's money, and
$z =$ C's money.

Then, (1) $x+y=600$,
(2) $x+z=700$,
and (3) $y+z=500$.
Adding (4) $2x+2y+2z=1800$.
(5) $x+y+z=900$.
$(5)-(1)=(6)$ $z=300$.
$(5)-(2)=(7)$ $y=200$.
$(5)-(3)=(8)$ $x=400$.
Answer, \$400, \$200, and \$300.

3. A number is expressed by 3 digits, the sum of which is 9; the number is 42 times the sum of the third and the second digits, and the first (units') digit is twice the sum of the other two: find the number.

NOTE.—Any number expressed by the French system of notation is equal to the sum of its 1st (units') digit, 10 times its 2d (tens') digit, 100 times its 3d (hundreds') digit, and so on. Convince yourself of this truth by applying it to several numbers.

Solution: Let $x =$ units' figure,
$y =$ tens' figure,
and $z =$ hundreds' figure.

Then, (1) $x+y+z=9$,
(2) $100z+10y+x=42(y+z)$,
and (3) $x=2(z+y)=2z+2y$.
$(2)=(4)$ $100z+10y+x=42y+42z$.
Col., (5) $x-32y+58z=0$.
$(5)-(1)=(6)$ $57z-33y=-9$.
Trans. in (3), (7) $x-2y-2z=0$.
$(1)-(7)=(8)$ $3y+3z=9$.
(9) $y+z=3$.
$\frac{1}{3}$ of $(6)=(10)$ $19z-11y=-3$.
$11\times(9)=(11)$ $11z+11y=33$.
$(10)+(11)=(12)$ $30z=30$.
(13) $z=1$.
$(9)=(14)$ $y+1=3$.
(15) $y=2$.
$(1)=(16)$ $x+2+1=9$.
(17) $x=6$.
Number, 126.

4. A, B, C, D and E have money; B gives A $\frac{1}{2}$ of his; C gives B $\frac{1}{3}$ of his; D gives C $\frac{1}{4}$ of his; and E gives D $\frac{1}{6}$ of his: each then has \$30. How much had each at first?

Solution: Let x = A's money,
y = B's money,
z = C's money,
w = D's money, and
r = E's money.

Then, (1) $x+\frac{1}{2}y=30$,
(2) $\frac{1}{2}y+\frac{1}{3}z=30$,
(3) $\frac{2}{3}z+\frac{1}{4}w=30$,
(4) $\frac{3}{4}w+\frac{1}{6}r=30$,
and (5) $\frac{5}{6}r=30$.
$\frac{6}{5}$ of (5) = (6) $r=36$.
(4) = (7) $\frac{3}{4}w+6=30$.
(8) $\frac{3}{4}w=24$.
(9) $w=32$.
(3) = (10) $\frac{2}{3}z+8=30$.
(11) $z=33$.
(2) = (12) $\frac{1}{2}y+11=30$.
(13) $y=38$.
(1) = (14) $x+19=30$.
(15) $x=11$.
Answer, \$11, \$38, \$33, \$32, \$36.

5. A and B can do a piece of work in 12 days; A and C in 15 days; and B and C in 20 days. In what time can each do it alone?

Solution: Let x = time req'd by A to do the work,
y = time req'd by B to do the work,
and z = time req'd by C to do the work.

Then, $\frac{1}{x}$ = part done by A in 1 da.
$\frac{1}{y}$ = part done by B in 1 da.
$\frac{1}{z}$ = part done by C in 1 da.

(1) $\frac{12}{x}+\frac{12}{y}=1$,

(2) $\frac{15}{x}+\frac{15}{z}=1$,

and (3) $\frac{20}{y}+\frac{20}{z}=1$.

$\frac{1}{15}$ of (2) = (4) $\frac{1}{x}+\frac{1}{z}=\frac{1}{15}$.

$\frac{1}{20}$ of (3) = (5) $\frac{1}{y}+\frac{1}{z}=\frac{1}{20}$.

$$\tfrac{1}{12} \text{ of } (1)=(6)\ \frac{1}{x}+\frac{1}{y}=\frac{1}{12}.$$

$$\text{Adding, } (7)\ \frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{12}{60}.$$

$$\tfrac{1}{2} \text{ of } 7=(8)\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{6}{60}=\frac{1}{10}.$$

$$(8)-(6)=(9)\ \frac{1}{z}=\frac{1}{60}, \text{ and } z=60.$$

$$(8)-(4)=(10)\ \frac{1}{y}=\frac{1}{30}, \text{ and } y=30.$$

$$(8)-(5)=(11)\ \frac{1}{x}=\frac{1}{20}, \text{ and } x=20.$$

Answer, 20 da., 30 da., and 60 da.

EXERCISE CLVIII.

1. Two calves, 5 sheep and 8 hogs cost $84; 8 calves, 6 sheep and 7 hogs cost $108; 10 calves, 12 sheep and 3 hogs cost $98: find the cost of a calf, a sheep, and a hog.

2. I bought corn, wheat and oats, as follows: 4 bushels of corn and 3 bushels of wheat for $3.70; 6 bushels of wheat and 8 bushels of oats for $6.20; 12 bushels of corn and 9 bushels of oats for $7.05: find the price of each per bushel.

3. A, B and C have $1000; A and B together own $800; B and C together own $630: what does each own?

4. A, B and C have $3600; if B gives A one-half of his, and C gives B one-third of his, they will all have the same amount: what had each at first?

5. After a battle in which 24000 men were engaged, it was found that the number slain was one-seventh of those who survived, and that the number wounded was equal to one-half of the slain: find the number slain, the number wounded, and the number not hurt.

6. A farmer has 45 head of horses, cows, and sheep; there are six times as many sheep as cows, and one-third as many horses as sheep: find the number of each.

7. A, B and C built a wall 200 feet long; B built as many feet as C and one third more, and A built three-fourths as much as B: how many feet did each build?

8. A man divided his estate of \$4000 as follows: To his wife \$500 more than to his son, and to the son \$1000 more than to the daughter: find the share of each.

9. Four persons compare their money; the first man has one-half as much as all the others; the second, one third as much as all the others; the third, one fourth as much as the other three; and the fourth, \$14 less than the first: find the amount of each man's money.

10. A says to B, "Give me \$100 and I will have as much money as you will have;" and he says to C, "Give me \$200 and I will have as much as you will have." B says to C, "Give me \$350 and my money will be to yours as 19: 9." Find how much each has.

11. A and B can do a piece of work in 48 days; A and C can do it in 30 days; B and C can do it in $26\frac{2}{3}$ days: in what time can each do it alone?

12. A, B and C earn \$63 in 14 days; A and B can earn it in 18 days; A and C can earn it in 21 days; in what time can each earn it alone?

13. A and B can earn \$40 in 6 days; A and C can earn \$54 in 9 days; B and C can earn \$80 in 15 days: what can each earn per day?

14. A cistern is filled by 3 pipes; the first and second fill it in 1 hr. 10 min.; the first and third in 1 hr. 24 min.; and the second and third in 2 hr. 20 min.: in what time can each fill it?

15. Find three numbers, such that the first together with $\frac{2}{3}$ of the second is equal to 19; $\frac{1}{3}$ of the second with $\frac{3}{4}$ of the third is equal to 23; and $\frac{1}{2}$ of the third with $\frac{1}{3}$ of the first is equal to the second.

16. A certain number is composed of three digits whose sum is 15. The digit in units' place is 3 times that in the hundreds'

place; and if 396 be added to the number, the digits will be reversed in order. Find the number.

17. A jeweler sold three rings. The price of the first with $\frac{1}{2}$ the sum of the second and third was \$25; the price of the second with $\frac{1}{3}$ of the sum of the first and third was \$26; and the price of the third with $\frac{1}{2}$ the sum of the first and second was \$29. What was the price of each?

18. Find four numbers such that, by adding each to twice the sum of the remaining three, you will obtain 46, 43, 41, and 38, respectively.

19. A gives to B and C twice as much money as each of them has; B then gives to A and C twice as much as each of them has; and C then gives to A and B twice as much as each of them has; each then has \$27. How much had each at first?

20. A, B, C and D together have \$40. After B gives to A $\frac{1}{3}$ of his money, C to B $\frac{1}{4}$ of his, and D to C $\frac{1}{5}$ of his, each has \$10. How much had each at first?

D. PROBLEMS CONTAINING QUADRATIC EQUATIONS.

179. Quadratic Equations.—An equation that contains the second power (or square) of the unknown number is called a **Quadratic Equation.** An equation that contains no form of the unknown number except the square is called an *incomplete* quadratic equation. An equation that contains both the first and second powers of the unknown number is called a *complete* quadratic equation. Thus:

(1) $x^2=64$. (Incomplete.)
(2) $x^2+4x=12$. (Complete.)

When an equation is of the form of (1), the value of the unknown number may be found by *extracting the square root of the equation.*

When an equation is of the form of (2), something must be

added to render the first member a *perfect square;* then, the value of the unknown number may be found by *extracting the square root of the equation.*

PRINCIPLE: *Extracting the square root of each member of an equation extracts the square root of the equation.*

EXAMPLES.

1. $6x^2=54$. Find the values of x.

Process: (1) $6x^2=54$.
$\frac{1}{6}$ of (1) = (2) $x^2=9$.
$\sqrt{(2)}$ = (3) $x=\pm 3$, result.

NOTE.—$+3\times+3=9$ and $-3\times-3=9$. Then, x may equal $+3$ or -3. When a number is true for either sign, the double sign $\pm$ is used. This sign is read *plus or minus*, and should be used whenever we extract the square root.

2. $\frac{x}{3}+\frac{3}{x}=\frac{x}{12}+\frac{12}{x}$. Find the values of x.

Process: (1) $\frac{x}{3}+\frac{3}{x}=\frac{x}{12}+\frac{12}{x}$.
$12x\times(1)$ = (2) $4x^2+36=x^2+144$.
(3) $4x^2-x^2=144-36$.
(4) $3x^2=108$.
(5) $x^2=36$.
$\sqrt{(5)}$ = (6) $x=\pm 6$, result.

We are now to study how to *complete the square* of the first member in a complete quadratic equation. We know that—

$$(x+a)^2=x^2+2ax+a^2, \text{ and}$$
$$(x-a)^2=x^2-2ax+a^2.$$

How could we get the third term in these results, if we had the second term?

Answer: Take half of the *2 a* and square it.

How can we get the third term in

$$x^2+12x \text{ and}$$
$$x^2-12x?$$

Answer: Take half of the *12* and square it. Thus:

$$x^2+12x+36$$
$$x^2-12x+36.$$

RULE FOR COMPLETING THE SQUARE: *Add the square of half the coefficient of the first power of the unknown number.*

3. $x^2-12x=13$. Find the values of x.

Process: (1) $x^2-12x=13$.
Completing the sq., (2) $x^2-12x+36=13+36$.
(3) $x^2-12x+36=49$.
$\sqrt{(3)}=$ (4) $x-6=\pm 7$.
(5) $x=6\pm 7=13$ or -1, results.

4. $5x^2+20x=60$. Find the values of x.

NOTE.—Always divide by the coefficient of x^2 before completing the square.

Process: (1) $5x^2+20x=60$.
(2) $x^2+4x=12$.
(3) $x^2+4x+4=12+4=16$.
$\sqrt{(3)}=$ (4) $x+2=\pm 4$.
(5) $x=2$ or -6.

5. $x^2+11x=-18$. Find the values of x.

Process: (1) $x^2+11x=-18$.
(2) $x^2+11x+\frac{121}{4}=-18+\frac{121}{4}=\frac{49}{4}$.
$\sqrt{(2)}=$ (3) $x+\frac{11}{2}=\pm\frac{7}{2}$.
(4) $x=-9$ or -1.

6. $\frac{x}{4}-\frac{44}{x-2}=4$. Find the values of x.

Process: (1) $\frac{x}{4}-\frac{44}{x-2}=4$.
(2) $x^2-2x-176=16x-32$.
(3) $x^2-2x-16x=176-32=144$.
(4) $x^2-18x=144$.
(5) $x^2-18x+81=144+81=225$.
$\sqrt{(5)}=$ (6) $x-9=\pm 15$.
$x=24$ or -6.

EXERCISE CLIX.

1. What is a *quadratic equation ?*

2. Define *incomplete* quadratic equation. *Complete* quadratic equation.

3. Give the principle governing extracting the square root of an equation.

4. How do you *complete the square* in a complete quadratic equation ?

Find the values of x *in—*

5. $x^2+2x=8.$

6. $x^2+6x=7.$

7. $x^2-10x=11.$

8. $x^2-8x=33.$

9. $x^2-x=\frac{3}{4}.$

10. $x^2+x=-\frac{1}{4}.$

11. $x^2-14x=-24.$

12. $x^2+5x=-2\frac{1}{4}.$

13. $x^2=625.$

14. $\frac{3x^2}{5}=15.$

15. $3x^2+7=43+2x^2.$

16. $\frac{7x^2}{2}=8+3x^2.$

17. $x^2+10x=11.$

18. $5x^2-40x=165.$

19. $4x^2-4x=3.$

20. $4x^2+20x=-9.$

21. $\frac{x+2}{x-1}-\frac{4-x}{2x}=\frac{7}{3}.$

180. Problems.

EXAMPLES.

1. A bought a lot of flour for \$126. If the number of \$'s per barrel was equal to $\frac{2}{7}$ of the number of barrels bought, find the number of barrels and the price per barrel.

Solution : Let x = number of barrels.

Then, $\frac{2}{7}x$ = price per barrel

and (1) $x\times\frac{2}{7}x=126.$

(2) $\frac{2}{7}x^2=126.$

(3) $x^2=441.$

$\sqrt{(3)}$ = (4) $x=\pm 21.$

(5) $\frac{2}{7}$ of $21=6.$

Answer, 21 bbls. @ \$6 each.

NOTE.—While the negative answer satisfies the equation it will not satisfy the problem, and the positive answer only is used in the problem.

2. Divide 12 into two parts, such that their product will be 11.

Solution: Let x = one part.
Then, $12-x$ = the other part
and (1) $x(12-x)=11$.
(2) $12x-x^2=11$.
(3) $x^2-12x=-11$.
(4) $x^2-12x+36=-11+36=25$.
(5) $x-6=\pm 5$.
(6) $x=11$, or 1.
(7) $12-x=1$, or 11.
Answer, 1 and 11.

3. If we should reduce the length of an edge of a cube 1 inch, the volume would be reduced 397 cu. in. What is the volume of the cube?

NOTE.—If we commence with the cube left and think of building the 1 inch back on to it, let x = one edge of that cube. Then, we must add a block x in. l., x in. w., 1 in. th., to each of three faces of the cube................ . $=3x^2$,
a rectangular solid x in. l., 1 in. w., 1 in. th., to each of three edges... $=3x$,
and a 1-inch cube in the corner........................ $=1$
The whole addition .... $=3x^2+3x+1$.

Then, (1) $3x^2+3x+1=397$.
(2) $3x^2+3x=397-1=396$.
(3) $x^2+x=132$.
(4) $x^2+x+\frac{1}{4}=132+\frac{1}{4}=\frac{529}{4}$.
$\sqrt{(4)}=$ (5) $x+\frac{1}{2}=\pm\frac{23}{2}$.
(6) $x=11,\ -12$.
The old cube 11 in. + 1 in. = 12 in., answer.

EXERCISE CLX.

1. Find two numbers, such that their difference is 5 and their product is 66.

2. Find that number whose square plus 6 times the number equals 55.

3. The sum of two numbers is 12, their product is 35. Find the numbers.

4. A rectangular field is twice as long as it is wide, and contains 20 acres. How wide is it?

5. The product of two consecutive integers exceeds six times their sum by 6. Find the numbers.

6. If we cut enough from a cubical block to make its dimensions 1 inch shorter, it will lose 1657 cu. in. Find the size of the block.

7. If we cut enough from a cubical block to make its dimensions 2 inches shorter, it will lose 728 cu. in. Find the size of the block.

8. The hypotenuse of a right triangle is 50 inches, its base equals one-half of its perpendicular. Find the base and perpendicular.

Hint: Let x = base.
Then, $2x$ = perpendicular,
and (1) $x^2+(2x)^2=50^2$, or
(2) $x^2+4x^2=2500$.

9. Find the altitude of an equilateral triangle if one side is 40 inches.

Hint: A perpendicular from the vertex to the base is the altitude. It divides the triangle into two equal right triangles. The base of each right triangle is one-half of the hypotenuse (40 in.). Letting x = the altitude, we have—

$$40^2=20^2+x^2.$$

10. A merchant sold a piece of cloth for $24, gaining as many % as the number of dollars the cloth cost. Find the cost.

11. A tree 90 ft. high was blown over by a storm so that the top touched the ground 40 ft. from the tree, while the other end of the part broken off rested on the stump. How much was broken off?

12. The base of a rectangle is 11 ft. longer than its altitude. If its area is 900 sq. ft., find the length of each side.

13. Find the altitude and area of a triangle whose sides are 15, 12, and 8.

NOTE.—When you drop the altitude on the base 15, you have two right triangles. Let x = the base of one triangle and $15-x$ the base of the other. Then,

$$(\text{Altitude})^2 = 12^2 - (15-x)^2 = 8^2 - x^2. \quad (\text{Why?})$$

Find the value of x, and complete the solution.

E. PROGRESSIONS.

181. Arithmetical Progressions.—An **Arithmetical Progression** is a series of numbers that increase or decrease by a **common difference.** Thus,

1, 4, 7, 10, 13,

is an *increasing* arithmetical progression. The common difference is 3.

11, 9, 7, 5, 3, 1,

is a *decreasing* arithmetical progression. The common difference is -2.

TERMS.

a, the first term.
l, the last term.
n, the number of terms.
d, the common difference.
S, the sum of the series.

Developing the formulas for arithmetical progressions.

I. *The formula for the last term:* Using the terms given above, we may write an arithmetical progression or series. Thus,

$$\begin{array}{llllll} (1) & (2) & (3) & (4) & (5) & \ldots\ldots\ldots (n) \\ S=a+ & (a+d)+ & (a+2d)+ & (a+3d)+ & (a+4d) & \ldots\ldots\ldots (\ ). \end{array}$$

Observe (1) that each term of the series has a, and that each term after the first has d; (2) that the coefficient of d is always *1 less than the number of the term.* Then, for the *nth* term, the coefficient of d is $(n-1)$.

From these data, we may write the last term. Thus,

$$l=a+(n-1)\,d.$$

RELATION: *In an arithmetical progression, the last term is equal to the first term, plus the common difference times the number of terms less 1.*

II. *The formula for the sum of the series:* An arithmetical series may be written thus:

(1) $S=a+(a+d)+(a+2d)\ldots\ldots\ldots+(l-2d)+(l-d)+l$;

also, (2) $S=l+(l-d)+(l-2d)\ldots\ldots\ldots+(a+2d)+(a+d)+a$.

Adding, (3) $2S=(a+l)+(a+l)+(a+l)\ldots\ldots\ldots+(a+l)+(a+l)+(a+l)$.

Since there are n terms and each has $(a+l)$,

(3)=(4) $2S=n(a+l)$; or,

$$S=\frac{n}{2}(a+l).$$

RELATION: *The sum of an arithmetical series is equal to the product of one-half of the number of terms and the sum of the extremes (the first and last terms).*

EXAMPLES.

1. Find the 7th term of the series, whose 1st term is 5, and the common difference is 9.

Solution: (1) $l=a+(n-1)d$.
(2) $l=5+(7-1)9=59$, answer.

2. Find the 10th term of the series 2, 6, 10, etc.

Solution: (1) $a=2$. (Why?)
(2) $d=4$. (Why?)
(3) $l=2+(10-1)4=38$, answer.

3. Find the sum of a series of 19 terms, whose first term is 6 and whose last term is 96.

Solution: (1) $S=\frac{n}{2}(a+l)$.

(2) $S=\frac{19}{2}(6+96)=969$, answer.

4. Find the last term and the sum of the series, *5*, *8*, etc., to *25* terms.

Solution: (1) $a=5$.
(2) $d=3$.
(3) $n=25$.
(4) $l=5+(25-1)3=77$, 1st answer.
(5) $S=\frac{25}{2}(5+77)=1025$, 2d answer.

5. Find the last term and the sum of the series *12*, *9*, *6*, etc., to *15* terms.

Solution: (1) $a=12$.
(2) $d=-3$. (Why?)
(3) $n=15$.
(4) $l=12+(15-1)(-3)$.
(5) $l=12-42=-30$, 1st answer.
(6) $S=\frac{15}{2}[12+(-30)]$.
(7) $S=\frac{15}{2}\times-18=-135$, 2d answer.

NOTE.—Be careful to handle the signs correctly, and a minus common difference will give you no trouble.

6. Find the first term of a series whose last term, common difference, and number of terms are respectively 27, 2, and 11.

Solution: (1) $27=a+(11-1)2=a+20$.
Trans., (2) $a=27-20=7$, answer.

7. Find the number of terms of the series whose first term, last term, and sum are respectively -1, 44, 215.

Solution: (1) $215=\frac{n}{2}(-1+44)=\frac{n}{2}\times 43$.

(2) $\frac{43n}{2}=215$.

(3) $n=\frac{215\times 2}{43}=10$, answer.

EXERCISE CLXI.

1. What is an arithmetical progression or series?

2. Name and explain the terms used.

3. Write a series of 8 terms whose first term is 5 and whose common difference is 12.

4. Write a series of 12 terms, whose first term is 12 and whose common difference is -5.

5. Develop the formula for finding l. Give the relation.

6. Develop the formula for finding S. Give the relation.

No.	l	a	n	d	S
7.	?	7	9	5	?
8.	?	-5	12	-3	?
9.	?	27	18	-4	?
10.	2	?	7	-1	?
11.	24	-15	14	?	?
12.	116	1	?	5	?
13.	31	?	7	7	?
14.	$13\frac{1}{2}$	$\frac{1}{2}$	27	?	?
15.	4	$-\frac{1}{5}$	?	$\frac{3}{5}$	?
16.	24	8	?	?	48
17.	?	27	9	?	63
18.	$5\frac{1}{2}$	?	12	?	-99
19.	$15\frac{1}{2}$	?	?	$\frac{3}{4}$	168
20.	?	?	17	$\frac{1}{3}$	$-39\frac{2}{3}$

NOTE.—The last two (Nos. 19 and 20) will require two equations. Substitute in both formulas. This will give you two equations to solve for two unknown numbers.

21. 40 potatoes are 2 yd. apart and the first is 2 yd. from a basket. How far will a boy travel, who gathers them and puts them into the basket one at a time?

182. Falling Bodies.—Falling of bodies furnishes an example of arithmetical series. A body, if left unsupported, will fall by its own weight during the 1st second, 16.08 ft. (N. Y.); during the 2d second, 48.24 ft.; during the 3d second, 80.4 ft.; and so on. Thus,

16.08, 48.24, 80.4, 112.56, ...

NOTE.—Observe that in this series the first term (a) is 16.08, and the *common difference* (d) is 32.16.

EXAMPLES.

1. How far will a body fall during the 10th second?

Solution: (1) $a=16.08$.
(2) $d=32.16$.
(3) $n=10$.
(4) $l=a+(n-1)d$.
(5) $l=16.08+9\times 32.16=305.52$.
Answer, 305.52 ft.

2. How far will a body fall in 8 seconds?

Solution: (1) $l=16.08+7\times 32.16=241.2$.
(2) $S=\frac{8}{2}(16.08+241.2)=1029.12$.
Answer, 1029.12 ft.

3. A body is thrown downward so that it travels 70 ft. the first second. How far does it fall in 5 seconds?

NOTE.— Here $a=70$.

Solution: (1) $l=70+4\times 32.16=198.64$.
(2) $S=\frac{5}{2}(70+198.64)=671.6$.
Answer, 671.6 ft.

In the falling of bodies it has been determined that the velocity increases 32.16 ft. per second. This is the *acceleration*. Using 32.16 for a in formula on page 309, we have—

$$V=32.16\,t.$$

RELATION: *Abstractly, the velocity of a falling body is 32.16 times the time expressed in seconds.*

4. A body strikes the ground with a velocity of 160.8 ft. per second. How many seconds has it been falling and how far has it fallen?

Solution: (1) $V = 32.16\,t$.

(2) $160.8 = 32.16\,t$.

(4) $t = \frac{160.8}{32.16} = 5$.

Time, 5 seconds.

(5) $l = 16.08 + 4 \times 32.16 = 144.72$.

(6) $S = \frac{5}{2}(16.08 + 144.72) = 402$.

Distance, 402 ft.

NOTE.—A body thrown upward will decrease in its travel just as a falling body increases in its travel.

5. How high will a body rise, if thrown upward with a velocity of 321.6 ft. per second?

Solution: (1) $V = 32.16\,t$.

(2) $321.6 = 32.16\,t$.

(3) $t = \frac{321.6}{32.16} = 10$.

Time in rising, 10 sec.

NOTE.—If it was 10 sec. rising, the last second it rose 16.08 ft. To find how far it rose the first second, consider from the top downward and get the 10th term of the series.

(4) $l = 16.08 + 9 \times 32.16 = 305.52$.

(5) $S = \frac{10}{2}(16.08 + 305.52) = 1608$.

Distance, 1608 ft.

EXERCISE CLXII.

1. How far will a body fall in 1 sec.? In 2 sec.? In 3 sec.?

2. Do the distances fallen during the successive seconds form an arithmetical progression? If so, what is the common difference?

3. What is the formula for the velocity of falling bodies?

4. A body thrown upward is gone 6 sec. How much time is taken in rising? How much in falling?

5. How fast will a body be falling at the end of the 6th second of its descent? Give its fall during the 7th second.

6. A boy throws a stone over a tree; it strikes the ground in 4 sec. How high is the tree?

7. A body strikes the ground with a velocity of 96.48 ft. per sec. How long has it been falling?

8. A body is thrown downward with a velocity of 60 ft. per second. What will its velocity be in 6 sec.? How far will it fall? ($a=60+16.08$. Why?)

9. A tower is 321.6 feet high. With what velocity will a stone, let fall from the top, strike the ground?

PLAN: (1) Find the time and (2) the velocity.

183. Geometrical Progressions.—A Geometrical Progression is a series of numbers that increase or decrease by a *common ratio.*

1, 4, 16, 64, etc.,

is an increasing geometrical series. The *common ratio* is 4.

24, 12, 6, 3, etc.,

is a decreasing geometrical series. The *common ratio* is $\frac{1}{2}$.

TERMS.

a, the first term.
l, the last term.
r, the common ratio.
n, the number of terms.
S, the sum of the series.

Developing the formulas for geometrical series.

I. *The formula for the last term:* Using the terms given above, we may write a geometrical series.

$$\begin{array}{l} \quad(1)\quad(2)\quad(3)\quad(4)\ldots\ldots\ldots\ldots\ldots(n). \\ S=a+ar+ar^2+ar^3\ldots\ldots\ldots\ldots\ldots(\ ). \end{array}$$

Observe (1) that each term of the series has a factor a, and (2) every term after the first has a factor r whose exponent is *1 less than the number of the term.* Then, we may write the last term—

$$l=ar^{n-1}$$

NOTE.—$n-1$ is the exponent of r.

RELATION: *In a geometrical series, the last term is equal to the first term times the ratio raised to a power whose exponent is 1 less than the number of terms.*

II. *The formula for the sum of the series:* The series may be written in this form:

$$\begin{array}{rl} & (1)\quad(2)\quad(3)\quad(4)\ldots\ldots\ldots\ldots(n) \\ (1) & S=a+ar+ar^2+ar^3\ldots\ldots\ldots\ldots l. \\ & \quad(1)\quad(2)\quad(3)\ldots\ldots\ldots\ldots(n-1)\quad(n) \\ r\times(1)=(2) & Sr=ar+ar^2+ar^3\ldots\ldots\ldots\ldots l\quad+lr \\ (2)-(1)=(3) & Sr-S=lr-a. \\ (4) & S(r-1)=lr-a;\text{ or,} \end{array}$$

$$S=\frac{lr-a}{r-1}.$$

NOTE.—When (1) is subtracted from (2), all the terms in the second members disappear, except the first term in (1) and the last term in (2).

RELATION: *The sum of a geometrical series is equal to the fraction whose numerator is the last term times the ratio, minus the first term, and whose denominator is the ratio less 1.*

EXAMPLES.

1. Find the fifth term of the series whose first term and common ratio are respectively 4 and 3.

Solution: (1) $l=ar^{n-1}$

(2) $l=4\times3^4=324$, answer.

2. Find the ratio of the series whose first term, number of terms and last term are respectively 20, 4, and 540.

Solution : (1) $l=ar^{n-1}$
(2) $540=20r^3$.
(3) $r^3=27$.
(4) $r=\sqrt[3]{27}=3$, answer.

3. Find the sum of 8 terms of the series, 3, 6, 12, etc.

Solution : (1) $r=2$.
(2) $l=3\times 2^7=384$.
(3) $S=\dfrac{rl-a}{r-1}$.
(4) $S=\dfrac{2\times 384-3}{1}=765$, answer.

4. Find the sum of the infinite series,

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\text{etc.}$$

Note.—An *infinite series* continues forever. This series is decreasing and its *last term* approaches infinitely near to 0—so near that no other value represents it so accurately as 0. Then, we have $a=1$, $r=\frac{1}{2}$, and $l=0$.

Solution : $S=\dfrac{0-1}{\frac{1}{2}-1}=\dfrac{-1}{-\frac{1}{2}}=2$, answer.

5. Find the number of terms in the series whose 1st term, last term, and ratio are respectively 5, 1280, 4.

Solution : (1) $1280=5\times 4^{n-1}$
(2) $4^{n-1}=256$.
(3) $4^4=256$.
(4) $n-1=4$.
(5) $n=5$, answer.

NOTE.—To obtain (3), you know that 4 raised to some power equals 256. Proceed by expanding 4, and when you reach the 4th power you will have 256.

EXERCISE CLXIII.

1. What is a geometrical progression or series?

2. Give and explain all terms used in geometrical series.

3. Write a geometrical series of 4 terms, whose 1st term is 5 and whose ratio is 7.

4. Write a series of 4 terms, whose 1st term is 4 and whose ratio is $\frac{1}{2}$.

5. Write 5 terms of the series whose 1st term is 4 and whose ratio is -3.

6. Develop the formula for l. Give the relation.

7. Develop the formula for S. Give the relation.

8. What is an infinite series? If the infinite series is decreasing, what does the l approach?

No.	l	a	n	r	S
9.	?	4	7	5	?
10.	?	-3	10	3	?
11.	?	1	8	-4	?
12.		2	infinite	$\frac{3}{5}$	?
13.	64	1	?	2	?
14.	?	4	?	3	118096
15.	13122	?	9	3	?
16.	?	2	8	$\frac{1}{2}$	?
17.	375000	?	7	5	?
18.		?	infinite	$\frac{1}{3}$	$7\frac{1}{2}$

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